CLASS 12

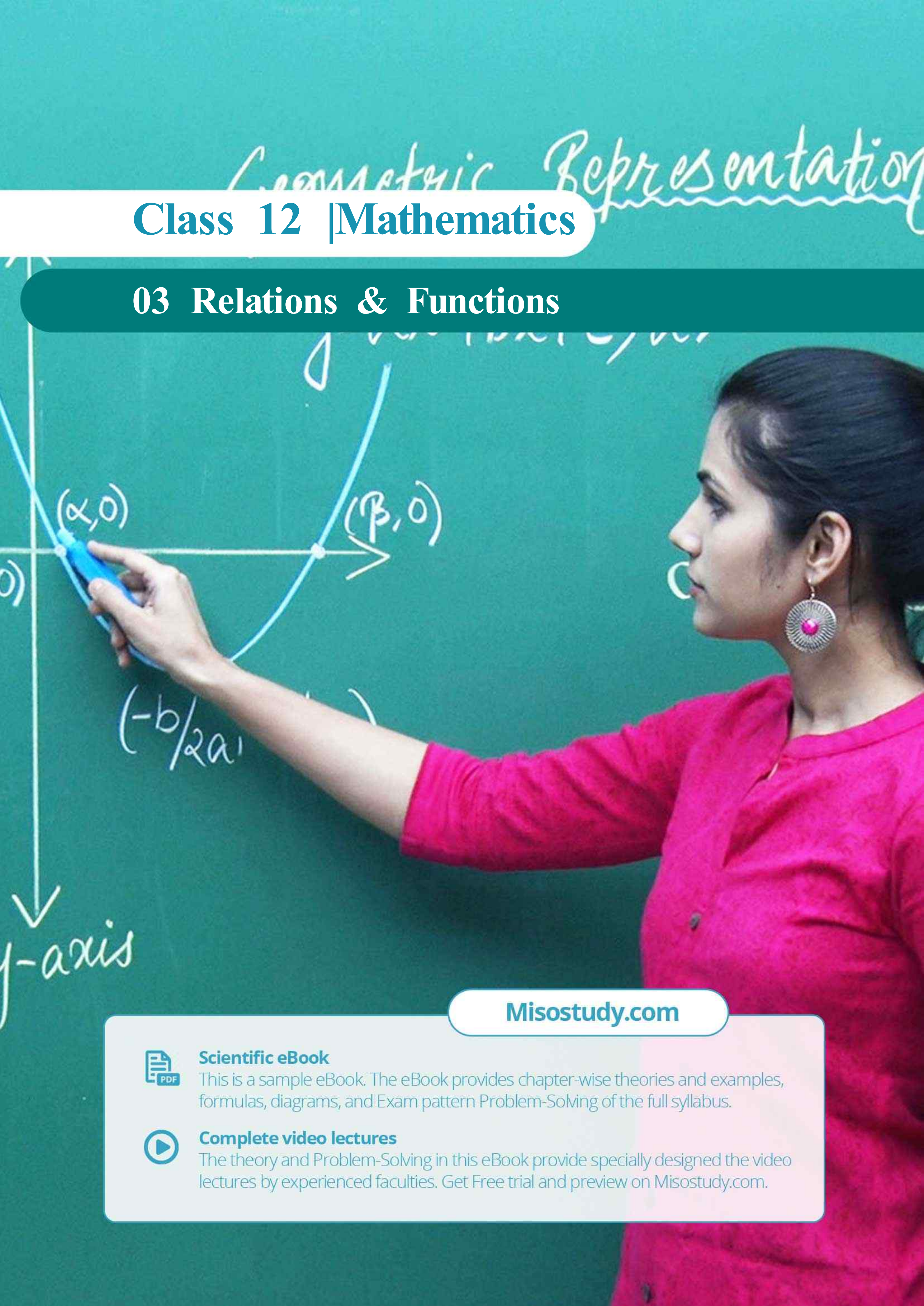
CBSE

MATHEMATICS

- Specially designed eBook for complete CBSE syllabus
- CBSE preparation strategy & direction, speedy revision
- Chapter-wise important Problem-Solving
- Theory, Formulas & Diagrams to crack CBSE
- Updated New Pattern Objective questions
- Detailed high quality video lectures by experienced faculties
- CBSE full preparation for Class XI & XII

misostudy
connect smart learning

JEE·NEET·AIIMS·CBSE·FOUNDATION
www.misostudy.com
☎ 8929803804 (MON-FRI : 9am-6pm) support@misostudy.com
MISO STUDY INDIA PVT. LTD.
2ND FLOOR 65-A, OMPRO TOWER, KALU SARAI, NEW DELHI, DELHI 110016



Class 12 |Mathematics

03 Relations & Functions

Misostudy.com

Scientific eBook
This is a sample eBook. The eBook provides chapter-wise theories and examples, formulas, diagrams, and Exam pattern Problem-Solving of the full syllabus.

Complete video lectures
The theory and Problem-Solving in this eBook provide specially designed the video lectures by experienced faculties. Get Free trial and preview on Misostudy.com.

01. Types of Relations

(A) **VOID, UNIVERSAL AND IDENTITY RELATIONS**

Void Relation-

Let A be a set. Then, $\phi \subseteq A\times A$ and so it is a relation on A. This relation is called the void or empty relation on set A.

Universal Relation-

Let A be a set. Then, $A\times A \subseteq A\times A$ and so it is a relation on A. This relation is called the universal relation on A.

NOTE ☞ It is to note here that the void relation and the universal relation on a set A are respectively the smallest and the largest relations on set A.
Both the empty (or void) relation and the universal relation are sometimes. They are called trivial relations.

Identity Relation-

Let A be a set. Then, the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation i.e., if every element of A is related to itself only.

(B) **REFLEXIVE, SYMMETRIC, TRANSITIVE RELATIONS**

Reflexive Relation-

A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Illustration I Let A = {1, 2, 3} be a set. Then R = {(1, 1), (2, 2), (3, 3), (1, 3), (2, 1) is a reflexive relation on A. But R_1 = {(1, 1), (3, 3), (2, 1), (3, 2)} is not a reflexive relation on A, because 2 $\in$ A but (2, 2) $\notin$ R_1.

Symmetric Relation-

A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.

Illustration I Let A = {1, 2, 3, 4} and let R_1 and R_2 be relations on A given by R_1 = {(1, 3), (1, 4), (3, 1), (2, 2) (4, 1)} and R_2 = {(1, 1), (2, 2), (3, 3), (1, 3)}. Clearly, R_1 is a symmetric relation on A. However, R_2 is not $\in$ R_2 but (3, 1) $\notin$ R_2.

NOTE ☞ A reflexive relation on a set A is not necessarily symmetric. For example, the relation R = {(1, 1), (2, 2), (3, 3), (1, 3) is a reflexive relation on set A = {1, 2, 3} but it is not symmetric.

Transitive Relation-

Let A be any set. A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $(a, c) \in R$ for all $a, b, c \in A$.

i.e. aRb and bRc

$\Rightarrow$ aRc for all $a, b, c \in A$.

Illustration I *On the set N of natural numbers, the relation R defined by xRy ⇒ x is less than y is transitive, because for any* $x, y, z \in N$ *$x < y$ and $y < z \Rightarrow x < z$ i.e., xRy and yRz ⇒ xRz*

(C) **Equivalence Relation**

A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Let R be a relation on the set of all line in a plane defined by $(l_1, l_2) \in R \Leftrightarrow$ line l_1 is parallel to line l_2.

Then, R is an equivalence relation.

Solution : Let L be the given set of all lines in a plane. Then we observe the following properties.

Reflexive : For each line $l \in L$, we have

$l \parallel l \Rightarrow (l, l) \in R$ for all $l \in L \Rightarrow R$ is reflexive

Symmetric : Let $l_1, l_2 \in L$ such that $(l_1, l_2) \in R$. Then,

$(l_1, l_2) \in R \Rightarrow l_1 \parallel l_2 \Rightarrow l_2 \parallel l_1 \Rightarrow (l_2, l_1) \in R$. So, R is symmetric on R.

Transitive : Let $l_1, l_2, l_3 \in L$ such that $(l_1, l_2) \in R$ and $(l_2, l_3) \in$ R. Then,

$(l_1, l_2) \in R$ and $(l_2, l_3) \in R \Rightarrow l_1 \parallel l_2$ and $l_2 \parallel l_3 \Rightarrow l_1 \parallel l_3 \Rightarrow (l_1, l_3) \in R$

So, R is transitive on L, Hence R being reflexive symmetric and transitive is an equivalence relation on L.

02. Kinds of Functions

ONE-ONE FUNCTION (INJECTION)

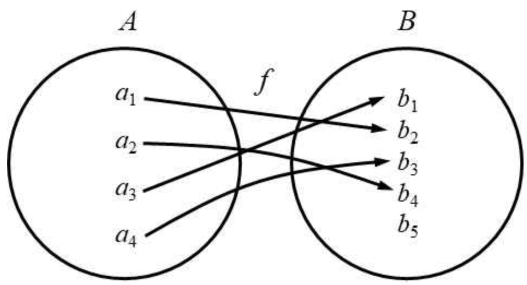
A function $f: A \to B$ is said to be a one-one function or an injection if different elements of A have different images in B.

Thus, $f: A \to B$ is one-one

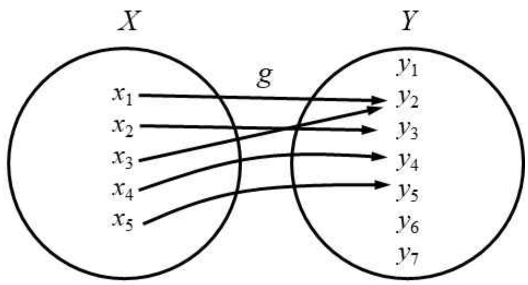
$\Leftrightarrow \quad a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$

$\Leftrightarrow \quad f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$

Example : Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams :



Figure



Figure

Clearly, $f : A \to B$ is a one-one function. But, $g : X \to Y$ is not one-one because two distinct elements x_1 and x_3 have the same image under function g.

Let $f : A \to B$ be a function such that A is an infinite set and we wish to check the injectivity of f. In such a case it is not possible to list the images of all elements of set A to see whether different elements of A have different images or not. The following algorithm provides a systematic procedure to check the injectivity of a function.

Algorithm

(i) Take two arbitrary elements x, y (say) in the domain of f.

(ii) Put $f(x) = f(y)$

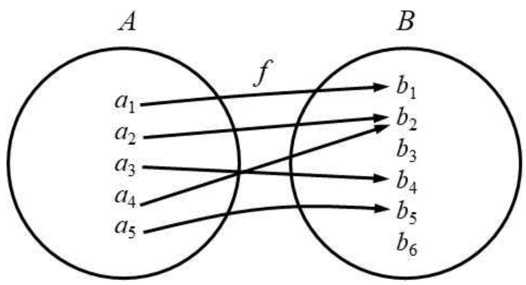
(iii) Solve $f(x) = f(y)$. If $f(x) = f(y$ gives x = y only, them $f: A \to B$ is a one-one function (or an injection). Otherwise not.

MANY-ONE FUNCTION

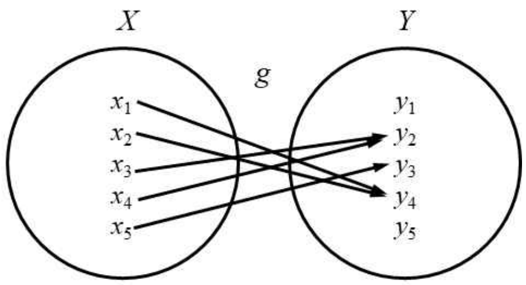
A function $f: A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

Thus, $f: A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

Example : Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams :



Figure



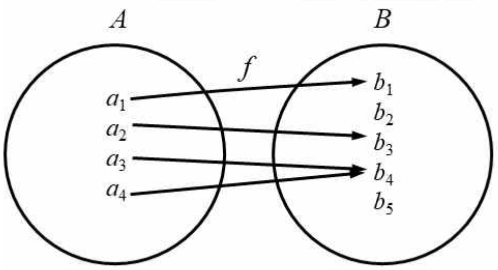
Figure

Clearly, $a_2 \neq a_4$ but $f(a_2) = f(a_4)$ and $x_1 \neq x_2$ but $g(x_1) = g(x_2)$.
So, f and g are many-one functions.

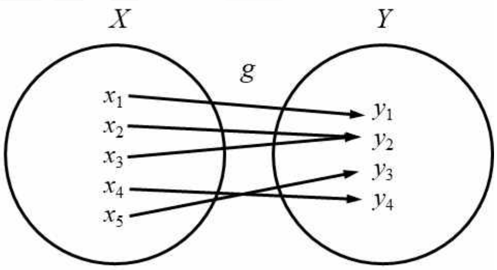
NOTE ☞ In other words, $f: A \rightarrow B$ is many-one function if it is not a one-one function.

ONTO FUNCTION (SURJECTION)
A function $f: A \rightarrow B$ is said to be an onto function or a surjection if every element of B is the f-image of some element of A i.e., if f(A) = B of range of f is the co-domain of f.
Thus, $f: A \rightarrow B$ is a surjection iff for each $b \in B$, there exists $a \in A$ such that f(a) = b.

INTO FUNCTION. A function $f: A \rightarrow B$ is an into function if there exists an element in B having no pre-image in A.
In other words, $f: A \rightarrow B$ is an into function if it is not an onto function.
Example : Let $f: A \rightarrow B$ and $g: X \rightarrow Y$ be two functions represented by the following diagrams :



Figure



Figure

Clearly, b_2 and b_5 are two elements in B which do not have their pre-images in A.
So, $f: A \rightarrow B$ is an into function.
Under function g every elements in Y has its pre-image X. So, $g: X \rightarrow Y$ is an onto function.
The following algorithm can be used to check the surjectivity of a real function.
Algorithm
Let $f: A \rightarrow B$ be the given function.
(i) Choose an arbitrary element y in B.
(ii) Put f(x) = y
(iii) Solve the equation f(x) = y for x and obtain x in terms of y. Let x = g(y).
(iv) If for all values of $y \in B$, the values of x obtained from x = g(y) are in A, then f is onto. If there are some $y \in B$ for which x, given by x = g(y), is not in A. Then, f is not onto.

BIJECTION (ONE-ONE ONTO FUNCTION)

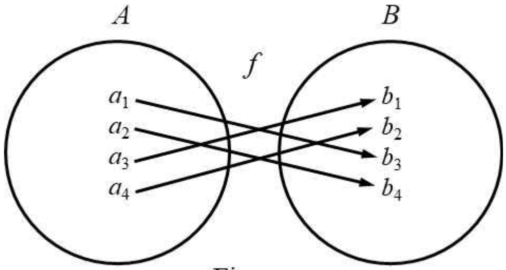
A function $f: A \to B$ is a bijection if it is one-one as well as onto. In other words, a function $f: A \to B$ is a bijection, if

(i) it is one-one i.e. $f(x) = f(y) \Rightarrow x = y$ for all x, $y \in A$.

(ii) it is onto i.e. for all $y \in B$, there exists $x \in A$ such that f(x) = y.

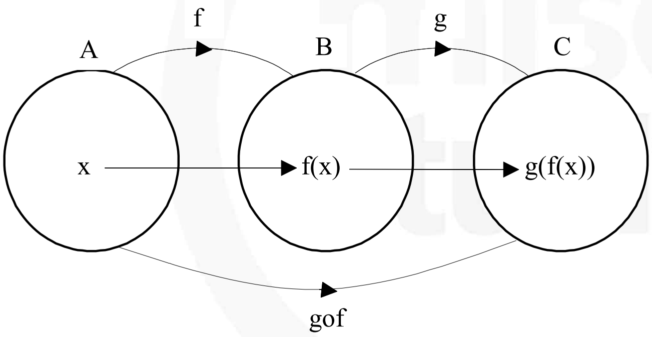
Example : *Let f : A → B be a function represented by the following diagram :*

Clearly, f is a bijection since it is both injective as well as surjective.



Figure

03. Composition of Functions



Definition:

Let $f: A \to B$ and $g: B \to C$ be two functions. Then a function go$f: A \to C$ defined by

$(gof)(x) = g(f(x))$, for all $x \in A$

is called the composition of f and g.

NOTE ☞ (i) It is evident from the definition that gof is defined only if for each $x \in A$, f(x) is an element of g so that we can take its g-image. Hence, for the composition gof to exist, the range of f must be subset of the domain of g.

(ii) It should be noted that gof exists iff the range of f is a subset of domain of g. Similarly, fog exists if range of g is a subset of domain of f.

PROPERTIES OF COMPOSITION OF FUNCTIONS

RESULT 1 The composition of functions is not commutative i.e. $fog \neq gof$.

RESULT 2 The composition of functions is associative i.e. *if f, g, h* are three functions such that *(fog)oh* and *fo(goh)* exist, then

$$(fog)oh = fo(goh)$$

Proof Let A, B, C, D be four non-void sets. Let $h : A \to B$, $g : B \to C$ and $f : C \to D$ be three functions. Then,

$h : A \to B,\ g : B \to C,\ f : C \to D$

$\Rightarrow$ $fog : B \to D$ and $h : A \to B$

Again, $h,\ A \to B,\ g : B \to C,\ f : C \to D$

$\Rightarrow$ $fog : C \to D$ and $goh : A \to C$

$\Rightarrow$ $fo\ (goh) : A \to D$

Thus, $(fog)oh$ and $fo(goh)$ are functions from set A to set D.

Now, we shall show that $\{(fog)oh\}\ (x) = \{fo\ (goh)\}\ (x)$ for all $x \in A$.

Let x be an arbitrary element of A and let $y \in B$, $z \in C$ such that $h\ (x) = y$ and $g(y) = z$. Then,

$h : A \to B,\ g : B \to C,\ f : C \to D$

$\Rightarrow$ $fog : B \to D$ and $h : A \to B$

Again, $h,\ A \to B,\ g : B \to C,\ f : C \to D$

$\Rightarrow$ $fog : C \to D$ and $goh : A \to C$

$\Rightarrow$ $fo\ (goh) : A \to D$

Thus, $(fog)oh$ and $fo(goh)$ are functions from set A to set D.

Now, we shall show that $\{(fog)oh\}\ (x) = \{fo\ (goh)\}\ (x)$ for all $x \in A$.

Let x be an arbitrary element of A and let $y \in B$, $z \in C$ such that $h\ (x) = y$ and $g(y) = z$. Then,

$\{(fog)\ oh\}\ (x) = (fog)\ \{h\ (x)\}$

$\Rightarrow$ $\{(fog)\ oh\}\ (x) = (fog)\ (y)$ $\quad [\because\ h(x) = y]$

$\Rightarrow$ $\{(fog)\ oh\}\ (x) = f(g\ (y))$

$\Rightarrow$ $\{(fog)\ oh\}\ (x) = f(z)$ $\quad$ ...(i) $\quad [\because\ g(y) = z]$

And, $\{fo\ (goh)\}\ (x) = f\{(goh)\ (x)\}$

$\Rightarrow$ $\{fo(goh)\}\ (x) = f\{g\ (h\ (x))\}$

$\Rightarrow$ $\{fo(goh)\}\ (x) = f\{g\ (y)\}$ $\quad [\because\ h(x) = y]$

$\Rightarrow$ $\{(fog)\ oh\}\ (x) = f(z)$ $\quad$ ...(ii) $\quad [\because\ g(y) = z]$

From (i) and (ii), we have

$\{(fog)\ oh\}\ (x) = \{fo\ (goh)\}\ (x)$ for all $x \in A$.

Hence, $(fog)\ oh = fo\ (goh)$

RESULT 3 The composition of two bijections is a bijection i.e. if f and g are two bijections, then gof is also a bijection.

Proof Let $f : A \to B$ and $g : B \to C$ be two bijections. Then, gof exists such that $gof : A \to C$.

We have to prove that gof is injective as well as surjective map.

Injectivity : Let x, y be two arbitrary elements of A. Then,

$\Rightarrow \quad (gof)(x) = (gof)(y)$

$\Rightarrow \quad g(f(x) = g(f(y))$

$\Rightarrow \quad f(x) = f(y)$ [$\because$ g is an injective map]

$\Rightarrow \quad x = y$ [$\because$ f is an injective map]

Thus, $(gof)(x) = (gof)(y)$ for all, $x, y \in A$

So, gof is an injective map.

Surjectivity : In order to prove the surjectivity of gof, we have to show that every element in C has its pre-image in A i.e. for all $z \in C$, there exists $x \in A$ such that $(gof)(x) = z$.

Let z be an arbitrary element of C. Then,

$z \in C \Rightarrow$ there exists $y \in B$ s.t.g. $(y) = z$ [$\because$ g is a surjective map]

and, $y \in B \Rightarrow$ there exists $x \in A$ s.t.f. $(x) = y$ [$\because$ f is a surjective map]

Thus, we find that for every $z \in C$, there exists $x \in A$ such that

$(gof)(x) = g(f(x)) = g(y) = z.$

i.e. every element of C is the gof-image of some element of A.

So, gof is a surjective map.

RESULT 4 Let $f : A \to B$. Then, $foI_A = I_Bof = f$ i.e. the composition of any function with the identity function is the function itself.

Proof Since $I_A : A \to A$ and $f : A \to B$, therefore, of $I_A : A \to B$. Now let x be an arbitrary element of A. Then,

$(fo\ I_A)(x) = f(I_A(x)) = f(x)$ [$\because$ $I_A(x) = x$ for all $x \in A$]

$\therefore \quad fo\ I_A = f$

Again, $f : A \to B$ and $I_B : B \Rightarrow I_B$ of $: A \to B$.

Now, let x be an arbitrary element of B. Let $f(x) = y$. Then, $y \in B$

$\therefore \quad (I_B\ of)(x) = I_B(f(x))$

$\Rightarrow \quad (I_B\ of)(x) = I_B(y)$ [$\because$ $f(x) = y$]

$\Rightarrow \quad (I_B\ of)(x) = y$

$\Rightarrow \quad (I_B\ of)(x) = f(x)$ [$\because$ $I_B(y) = y$ for all $y \in B$]

$\therefore \quad I_B\ of = f$

Hence, $fo\ I_A = I_B\ of = f$

RESULT 5 Let $f : A \to B$, $g : B \to A$ be two functions such that $gof = I_A$. Then, f is an injection and g is a surjection.

Proof *f is an injection* : Let $x, y \in A$ such that $f(x) = f(y)$. Then,

$f(x) = f(y)$

$\Rightarrow$ $g(f(x)) = g(f(y))$

$\Rightarrow$ $gof(x) = gof(y)$

$\Rightarrow$ $I_A(x) = I_A(y)$ [$\because gof = I_A$ (Given)]

$\Rightarrow$ $x = y$ [By def. of I_A]

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$

So, f is an injective map.

g is a surjection : We have, $g : B \to A$. In order to prove that g is a surjection. It is sufficient to prove that every element in A has its pre-image in B.

Let x be an arbitrary element of A. Then, as $f : A \to B$ is a function therefore $f(x) \in B$.

Let $f(x) = y$. Then,

$g(y) = g(f(x))$

$\Rightarrow$ $g(y) = gof(x)$

$\Rightarrow$ $g(y) = I_A(x)$ [$\because gof = I_A$]

$\Rightarrow$ $g(y) = x$

Thus, for every $x \in A$ there exists $y = f(x) \in B$ such that $g(y) = x$.

So, g is a surjection.

RESULT 6 Let $f : A \to B$ and $g : B \to A$ be two function such that $fog = I_B$. Then, f is a surjection and g is an injection.

Proof *f is a surjection* : In order to prove that $f : A \to B$ is a surjection, it is sufficient to prove that every element in B has its pre-image in A.

Let b be an arbitrary element of B.

Since g : B $\to$ A. Therefore, $g(b) \in A$.

Let $g(b) = a$.

Now, $f(a) = f(g(b))$ [$\because a = g(b)$]

$\Rightarrow$ $f(a) = fog(b)$

$\Rightarrow$ $f(a) = I_B(b)$ [$\because fog = I_B$]

$\Rightarrow$ $f(a) = b$

Thus, for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

So, f is a surjection.

g is an injection : Let x, y be any two elements of B such that $g(x) = g(y)$. Then,

$$g(x) = g(y)$$

$$\Rightarrow \quad f(g(x)) = f(g(y))$$

$$\Rightarrow \quad fog(x) = fog(y)$$

$$\Rightarrow \quad I_B(x) = I_B(y)$$

$$\Rightarrow \quad x = y$$

Thus, $g(x) = g(y) \Rightarrow x = y$ for all $x, y, \in B$.

So, g is an injection.

RESULT 7 Let $f : A \to B$ and $g : B \to A$ be two functions. Then,

(i) $gof : A \to C$ is into $\Rightarrow g : B \to C$ is onto

Proof In order to prove that $g : B \to C$ is onto whenever gof : $A \to$ is onto, it is sufficient to prove that for all $z \in C$ there exists $y \in B$ such that $g(y) = z$.

Let z be an arbitrary element of C. Since gof : $A \to C$ is onto. Therefore, there exists $x \in A$ such that

$$gof(x) = z$$

$$\Rightarrow \quad g(f(x)) = z$$

$$\Rightarrow \quad g(y) = z, \text{ where } y = f(x) \in B.$$

Thus, for all $z \in C$, there exists $y = f(x) \in B$ such that $g(y) = z$.

Hence, $g : B \to C$ is onto.

(ii) $gof : A \to C$ is one-one $\Rightarrow f : A \to B$ is one-one.

Proof In order to prove that $f : A \to B$ is one-one, it is sufficient to prove that $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

Let $x, y \in A$ such that $f(x) = f(y)$. Then,

$$f(x) = f(y)$$

$$\Rightarrow \quad g = (f(x)) = g(f(y)) \quad [\because g : B \to \text{C is a function}]$$

$$\Rightarrow \quad gof(x) = gof(y)$$

$$\Rightarrow \quad x = y \quad [\because gof : A \to \text{C is a function}]$$

Hence, $f : A \to B$ is one-one.

(iii) $gof : A \rightarrow C$ is onto and $\Rightarrow g : B \rightarrow C$ is one-one $\Rightarrow f : A \rightarrow B$ is onto.

Proof In order to prove that $f : A \rightarrow B$ is onto, it is sufficient to prove that for all $y \in B$ there exists $x \in A$ such that $f(x) = y$.

Let y be an arbitrary element of B. Then,

$g(y) \in C$ [$\because g : B \rightarrow$ C]

Since $gof : A \rightarrow C$ is an onto function. Therefore, for any $g(y) \in C$ there exists $x \in A$ such that $go\, f(x) = g(y)$

$\Rightarrow \quad g = (f(x)) = g(y)$

$\Rightarrow \quad f(x) = y$ [$\because g$ is one-one]

Thus, for all $y \in b$ there exists $x \in A$ such that $f(x) = y$.

Hence, $f : A \rightarrow B$ is onto.

(iv) $gof : A \rightarrow C$ is one-one and $\Rightarrow f : A \rightarrow B$ is onto $\Rightarrow g : B \rightarrow C$ is one-one.

Proof Let $y_1, y_2, \in B$ such that $g(y_1) = g(y_2)$. In order to prove that g is one-one, it is sufficient to prove that $y_1 = y_2$.

Now, $f : A \rightarrow B$ is onto and $y_1, y_2, \in B$. So, there exist $x_1, x_2 \in A$ such that

$f(x_1) = y_1$ and $f(x_2) = y_2$

Now, $g(y_1) = g(y_2)$

$\Rightarrow \quad g(f(x_1)) = g(f(x_2))$

$\Rightarrow \quad gof(x_1) = gof(x_2)$

$\Rightarrow \quad x_1 = x_2$ [$\because gof : A \rightarrow C$ is one-one]

$\Rightarrow \quad f(x_1) = (x_2)$ [$\because f : A \rightarrow$ B is a function]

$\Rightarrow \quad y_1 = y_2$

Hence, $g : B \rightarrow C$ is one-one.

04. Inverse of An Element

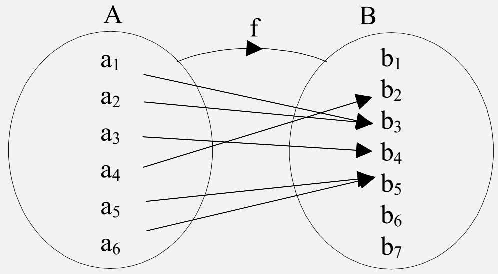
Let A and B be two sets and let $f: A \rightarrow B$ be a mapping. If $a \in A$ is associated to $b \in B$ under the function f, then $'b'$ is called the f image of $'a'$ and we write it as $b = f(a)$. We also say that $'a'$ is the pre-image or inverse element of $'b'$ under f and we write $a = f^{-1}(b)$.

NOTE ☞ The inverse of an element under a function may consist of a single element, two or more elements or no element depending on whether function is injective or many-one; onto or into.

If f is represented by Figure, then we find that

$$f^{-1}(b_1)=\phi, f^{-1}(b_2)=a_4,$$
$$f^{-1}(b_3)=\{a_1,a_2\}, f^{-1}(b_4)=a_3,$$
$$f^{-1}(b_5)=\{a_5,a_6\}, f^{-1}(b_6)=\phi$$

and, $f^{-1}(b_7)=\phi$



05. Inverse of A Function

Definition:

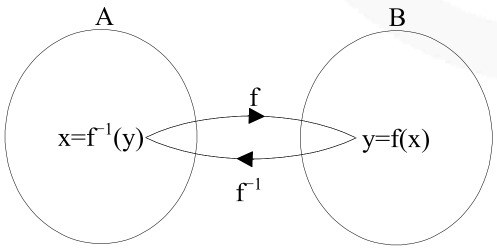
Let $f:A\to B$ be a bijection. Then a function $g:B\to A$ which associates each element $y\in B$ to a unique element $x\in A$ such that $f(x)=y$ is called the inverse of f.

i.e., $f(x)=y\Leftrightarrow g(y)=x$

The inverse of f is generally denoted by f^{-1}

Thus, if $f:A\to B$ is a bijection, then $f^{-1}:B\to A$ is such that

$$f(x)=y\Leftrightarrow f^{-1}(y)=x$$



In order to find the inverse of a bijection, we may follow the following algorithm.

Algorithm

Let $f : A \to B$ be a bijection. To find the inverse of f we follow the following steps:

STEP I Put $f(x) = y$, where $y \in B$ and $x \in A$.

STEP II Solve $f(x) = y$ to obtain x in terms of y.

STEP III In the relation obtained in step II replace x by $f^{-1}(y)$ to obtain the required inverse of f.

06. Properties of Inverse of a Function

RESULT 1 If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity function on the sets A and B respectively.

Proof In order to prove that gof $= I_A$ and $fog = I_B$, we have to prove that $(gof)(x) = x$ for all $x \in A$ and $(fog)(y) = y$ for all $y \in B$.

Let x be an element of A such that $f(x) = y$. Then,

$g(y) = x$ [$\because$ g is inverse of f]

Now, $(gof)(x) = g(f(x)) = g(y) = x$

$(gof)(x) = x$ for all $x \in A$

$\Rightarrow$ $gof = I_A$.

We have,

$(fog)(y) = f(g(y)) = f(x) = y$

$\therefore$ $fog(y) = y$ for all $y \in B$

$\Rightarrow$ $fog = I_B$.

Hence, $gof = I_A$ and fog $= I_B$.

RESULT 2 If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is a bijection and $(gof)^{-1} = f^{-1} o g^{-1}$

Proof We have,

$$\left.\begin{array}{l} f : A \to B \text{ is a bijection} \\ g : B \to C \text{ is a bijection} \end{array}\right\} \Rightarrow gof : A \to C \text{ is a bijection} \Rightarrow (gof)^{-1} : C \to A \text{ exists.}$$

Again,

$$\left.\begin{array}{l} f : A \to B \text{ is a bijection} \Rightarrow f^{-1};\ B \to A \text{ is a bijection} \\ g : B \to C \text{ is a bijection} \Rightarrow f^{-1};\ B \to A \text{ is a bijection} \end{array}\right\} \Rightarrow f^{-1} og^{-1} : C \to A$$

Let $x \in A$, $y \in B$ and $z \in C$ such that $f(x) = y$ and $g(y) = z$. Then,

$(gof)(x) = g(f(x)) = g(y) = z$

$\Rightarrow$ $(gof)^{-1}(z) = x$ ...(i)

Now,

$f(x) = y$ and $g(y) = z$

$\Rightarrow$ $f^{-1}(y) = x$ and $g^{-1}(z) = y$

$\therefore$ $(f^{-1} og^{-1})(z) = (f^{-1}(g^{-1}(z)) = f^{-1}(y) = x$ ...(ii)

From (i) and (ii), we have

$(gof)^{-1}(z) = (f^{-1} og^{-1})(z)$ for all $z \in C$.

Hence, $(gof)^{-1} = f^{-1} og^{-1}$.

RESULT 3 If $f: A \rightarrow B$ and $g: B \rightarrow A$ be two functions such that $gof = I_A$ and $fog = I_B$. Then, f and g are bijections and $g = f^{-1}$.

Proof f *is one-one* : Let $x, y \in A$ such that $f(x) = f(y)$. Then,

$f(x) = f(y)$

$\Rightarrow$ $g(f(x)) = g(f(y))$

$\Rightarrow$ $(gof)(x) = (gof)(y)$

$\Rightarrow$ $I_A(x) = I_A(y)$

$\Rightarrow$ $x = y$

$\therefore$ f is a one-one map.

f is onto : Let $y \in B$ and let $g(y) = x$. Then,

$g(y) = x$

$\Rightarrow$ $f(g(y)) = f(x)$

$\Rightarrow$ $(fog)(y) = f(x)$

$\Rightarrow$ $I_B(y) = f(x)$ $[\because fog = I_B]$

$\Rightarrow$ $y = f(x)$ $[\because I_B(y) = y]$

Thus, for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

So, f is onto.

Hence, f is a bijection.

Similarly, it can be proved that g is a bijection.

Now we shall show that $g = f^{-1}$.

Since $f : A \rightarrow B$ is a bijection. Therefore, f^{-1} exists.

Now, $fog = I_B$

$\Rightarrow$ $f^{-1} 0 (fog) = f^{-1} 0\, I_B$

$\Rightarrow$ $(f^{-1} of) og = f^{-1} 0\, I_B$ [By associativity]

$\Rightarrow$ $I_A\, 0\, g = f^{-1}\, 0\, I_B$ $[\because f^{-1} of\ I_A]$

$\Rightarrow$ $g = f^{-1}$ $[\because I_A og = g \text{ and } f^{-1} 0\, I_B = f^{-1}]$

Hence, $g = f^{-1}$

07. Binary Operation

DEFINITION

A binary operation * on a set A is a function * : A × A → A. We denote * (a, b) by a * b.

Eg. : Addition, subtraction and multiplication are binary operations on **R**, but division is not a binary operation on **R.** Further division is a binary operation on the set **R**. of non-zero real numbers.

Solution + : **R** × **R** → **R** is given by
$(a, b) \to a + b$
– : **R** × **R** → **R** is given by
$(a, b) \to a - b$
× : **R** × **R** → **R** is given by
$(a, b) \to ab$

Since '+', '–' and '×' are functions, they are binary operation on **R.**

But ÷ : **R** × **R** → **R,** given by $(a, b) \to \frac{a}{b}$ is a function and hence a binary operation on **R**.

Types of Binary Operation

Type I **Commutativity** *A binary operation '*' on a set S is said to be a communicative binary operation, if*

$$a * b = b * a \text{ for all } a, b \in S$$

Eg.: The binary operations addition (+) and multiplication (×) are commutative binary operation on Z. However, the binary operation subtraction (–) is not a commutative binary operation on Z as $3 - 2 \neq 2 - 3$.

Type II **Associativity** *A binary operation '*' on a set S is said to be an associative binary operation, if*

$$(a * b) * c = a * (b * c) \text{ for all } a, b \in S.$$

Eg.: The binary operations of addition (+) and multiplication (×) are associative binary operation on Z. However, the binary operation subtraction (–) is not a associative binary operation on Z as $(2 - 3) - 5 \neq 2 - (3 - 5)$.

Identity Element

Let "' be a binary operation on a set S. If there exists an elements $e \in S$ such that*

$$a*e = a = e * a \text{ for all } a \in S.$$

Then, e is called an identity element for the binary operation '' on set S.*

Eg. : Consider the binary operation of addition (+) on Z. we know that $0 \in Z$ such that

$$a + 0 = a = 0 + a \text{ for all } a \in Z$$

So, '0' is the identity element for addition on Z.

If we consider multiplication on Z, then '1' is the identity element for multiplication on Z, because

$$1 \times a = a = a \times 1 \text{ for all } a \in Z.$$

Eg. : We know that addition (+) and multiplication (·) are binary operations on N such that

$$n \times 1 = n = 1 \times n \text{ for all } n \in N$$

But, there do not exists any natural number e such that

$$n + e = n = e + n \text{ for all } n \in N.$$

So, 1 is the identity element for multiplication on N, but N does not have identity element for addition on N.

Inverse of an Element

Invertible Element *Let '*' be a binary operation on a set S, and let e be the identity element in S for this binary operation * on S. Then, an element a ∈ S is called an invertible element if there exists an element b ∈ S such that*

$$a * b = e = b * a$$

The element b is called an inverse of element a.

Thus, an element b ∈ S is called an inverse of an element a ∈ S, if

$$a * b = e = b * a.$$

Eg.: Consider the binary operation addition (+) on Z. Clearly, 0 is the identity element for addition on Z and for any integer a, we have

$$a + (-a) = 0 = (-a) + a$$

So, $-a$ is the inverse of $a \in Z$.

Multiplication is also a binary operation on Z and 1 is the identity element for multiplication on Z. But, no element, other than $1 \in Z$, is invertible.

Composition Table

A binary operation on finite set can be completely describe by means of a table known as a composition table. Let $S = \{a_1, a_2, ..., a_n\}$ be a finite set and * be a binary operation on S. Then the composition table for * is constructed in the manner indicated below.

We write the elements $a_1, a_2, ..., a_n$ of the set S in the top horizontal row and the left vertical column in the same order. Then we put down the element $a_i * a_j$ at the intersection of the row headed by a_i $(1 \le j \le n)$ to get the following table :

*	a_1	a_2	...	a_i	...	a_j	...	a_n
a_1	$a_1 * a_1$	$a_1 * a_2$	...	$a_i * a_i$	...	$a_1 * a_j$	...	$a_1 * a_n$
a_2	$a_2 * a_1$	$a_2 * a_2$	...	$a_2 * a_1$	...	$a_2 * a_j$	...	$a_2 * a_n$
$\vdots$								
a_i	$a_i * a_1$	$a_i * a_2$	...	$a_i * a_i$	...	$a_i * a_j$	...	$a_i * a_n$
$\vdots$								
a_j	$a_j * a_1$	$a_j * a_2$	...	$a_j * a_i$	...	$a_j * a_j$	...	$a_j * a_n$
$\vdots$								
a_n	$a_n * a_1$	$a_n * a_2$	...	$a_n * a_i$	...	$a_n * a_j$	...	$a_n * a_n$

(i) From the composition table we infer the following results.
If all the entires of the table are elements of set S and each element of S appears once and only once in each row and in each column, then the operation is a binary operation.

Sometimes we also say that the binary operation is well defined which means that the operation*. But for us, this is a consequence of the definition of binary operation.

(ii) If the entries in the table are symmetric with respect to the diagonal which starts at the upper left corner of the table and terminates at the lower right corner, we say that the binary operation is commutative on S, otherwise it is said to be non-commutative on S.

(iii) If the row headed by an element say a_j coincides with the row at the top and the column headed by a_j coincides with the column on extreme left, then a_j is the identity element in S for the binary operation *on S.

(iv) If each row except the top most row or each column except the left most column contains the identity element every element of S is invertible with respect to *. To find the inverse of an element say a_i, we consider row (or column) headed by a_i. Then we determine the position of identity element e in this row (or column). If e appears in the column (or row) headed by a_j, then a_i and a_j are inverse of each other.

CBSE Exam Pattern Exercise
Subjective Questions (1)

(Q 1 to 4) One Mark

1. Let R is the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : 2 \text{ divides } (a - b)\}$. Write the equivalence class [0].

2. If $f : R \to R$ is defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$.

3. Let $* : R \times R \to$ given by $(a, b) \to a + 4b^2$ be a binary operation. Compute $(-5) * (2 * 0)$.

4. If $*$ is a binary operation on set Q of rational number defined as $a * b = \frac{ab}{5}$. Write the identity for $*$, if any.

(Q 5 to 8) Four Marks

5. If the function $f : R \to R$ is given by $f(x) = x^2 + 2$ and $g : R \to R$ is given by $g(x) = \frac{x}{x-1}$; $x \neq 1$, then find fog and gof, and hence find fog (2) and gof (–3).

6. Show that $f : N \to N$, given by $f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$ is bijective (both one-one and onto).

7. Show that the relation S in the set R of real numbers defined as $S = \{(a, b) : a, b \in R \text{ and } a \leq b^3\}$ is neither reflexive nor symmetric nor transitive.

8. Consider the binary operation $*$ on the set $\{1, 2, 3, 4, 5\}$ defined by $a * b = \min \{a, b\}$. Write operation table of operation $*$.

(Q 9 to 10) Six Marks

9. Let $f : N \to R$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \to S$, where S is the range of f, is invertible. Also, find the inverse of f.

10. Let $A = Q \times Q$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for $(a, b)\ (c, b) \in A$. Determine, whether $*$ is commutative and associative. Then, with respect to

* on A

(i) Find the identity element in A.

(ii) Find the invertible elements of A.

Answer & Solution

Q1.

[0] $= \{b \in A;\ bR0\}$
$= \{b \in A;\ (b, 0) \in R\}$
$= \{b \in A;\ 2 \text{ divides } b - 0\}$
$= \{0, 2, 4\}$

Q2.

fof(x) $= f((3 - x)^{1/3})$
$= (3 - y^3)^{1/3}$ where $y = (3 - x^3)^{1/3}$
$= (3 - ((3 - x^3)^{1/3})^3)^{1/3}$
$= (3 - (3 - x^3))^{1/3} = (x^3)^{1/3} = x = I(x)$
$\therefore$ $fof = I_{\mathbf{R}}$

Q3.

$a * b = a + 4b^2$
$\Rightarrow -5 * (2 * 0)$
$\Rightarrow -5 * (2 + 4(0^2))$
$= -5 * 2$
$= -5 + 4(2)^2$
$= -5 + 16 = 11$

Q4.

Let e be the identity element in Q under *
$\Rightarrow a * e = a \ \forall \ a \in Q$
$= e * a$
$\Rightarrow \dfrac{ae}{5} = a \ \forall \ a \in Q$
$\Rightarrow e = 5 \in Q$
$\therefore$ Identity element = 5

Q5.

$f(x) = x^2 + 2 \ \forall \ x \in \text{IR}$
$D(f) = \text{IR}$
$R(f) = [2, \infty]$

($\because$ $x^2 \geq 0$ $\forall$ $x \in \mathbf{R} \Rightarrow x^2 + 2 \geq 2$ $\forall$ $x \in \mathbf{R}$ $\therefore$ $x^2 + 2 \in [2, \infty)$)

$g(x) = \frac{x}{x-1}$ & $g(x) \not\in \mathbf{R}$ for $x = 1$

$\therefore$ D(g) = $\mathbf{R}$/{1}

& R(g) = $\mathbf{R}$/{1}.

$$\left(\frac{x}{x-1} = y \Rightarrow x = yx - y \right.$$
$$\left. \Rightarrow x = \frac{y}{y-1} \in R \,\forall\, y \in R/\{1\} \right)$$

As R(g) $\subseteq$ D(f)

$\therefore$ fog exists

for $x \in D(g) = \mathbf{R} - \{1\}$,

$(fog)(x) = f(g(x))$

$$= f\left(\frac{x}{x-1}\right)$$

$$= \left(\frac{x}{x-1}\right)^2 + 2$$

$$= \frac{x^2}{(x+1)^2} + 2$$

$$= \frac{3x^2 - 4x + 2}{(x+1)^2}$$

$$\therefore fog(2) = \frac{3(2)^2 - 4(2) + 2}{(2-1)^2}$$

$$= 6$$

And, as $R(g) \subseteq D(f)$

$\therefore$ gof is defined

$\therefore$ for $x \in \mathbf{R} = D(f)$

$gof(x) = g(f(x))$

$$= g(x^2 + 2) = \frac{x^2 + 2}{x^2 + 2 - 1}$$

$$= \frac{x^2 + 2}{x^2 + 1}$$

$$\therefore gof(-3) = \frac{(-3)^2 + 2}{(-3)^2 + 1} = \frac{11}{10}$$

Q6.

Injective :

case (i) if x_1 & x_2 both are odd.

Then, $f(x_1) = f(x_2)$

$\Rightarrow x_1 + 1 = x_2 + 1$

$\Rightarrow x_1 = x_2$

case (ii) if x_1 & x_2 both are even

Then, $f(x_1) = f(x_2)$

$\Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2$

case (iii) If x_1 is even & x_2 is odd

$\Rightarrow x_1 \neq x_2$

to show $f(x_1) \neq f(x_2)$

As $x_1 \neq x_2 \Rightarrow x_1 - 1 \neq x_2 + 1$

$\Rightarrow f(x_1) \neq f(x_2)$

$\therefore$ From all three cases f is one-one

Surjective :

let $y \in \mathbf{N} = C(f)$

If y is odd then $y + 1$ is an even natural no ;

$f(y + 1) = (y + 1) - 1 = y$

$\therefore \exists x = y + 1 \in \mathbf{N}$ such that

$f(x) = y \in \mathbf{N}$

And, if y is even then $y - 1$ is an odd natural no;

$f(y - 1) = (y - 1) + 1 = y$

$\therefore \exists x = y - 1 \in \mathbf{N}; f(x) = y \in \mathbf{N}$

$\therefore f$ is onto.

Q7.

Reflexive :

Claim : $1/2 \not{R}\ 1/2$

$\because \frac{1}{2} \not\leq \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$\therefore 1/2 \not{R}\ 1/2$

$\therefore (1/2, 1/2) \not\in R$

$\therefore$ R is not reflexive.

Symmetric :

Claim : $-2R3$ but $3\not{R} -2$

As $-2 \leq 3^3 \Rightarrow (-2, 3) \in R$

But $3 \not\leq (-2)^3 = 8 \therefore$

$(3, -2) \not\in R$

$\therefore$ R is not symmetric.

Transitive :

Claim : $2R3/2$ & $3/2\ R4/3$ but $2 \not{R}\ 4/3$

As $2 \leq (3/2)^3$ & $3/2 \leq (4/3)^3$ but $2 \not\leq (4/3)^3$

$\Rightarrow (2, 3/2) \in R, (3/2, 4/3) \in R$ but $(2, 4/3) \not\in R$

$\therefore$ R is not transitive.

Q8.

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Q9.

ONE ONE :

Let $f(x_1) = f(x_2)$

$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow 9x_1^2 - 9x_2^2 + 6x_1 - 6x_2 = 0$

$9(x_1 + x_2)(x_1 - x_2) + 6(x_1 - x_2) = 0$

$(x_1 - x_2)(3x_1 + 3x_2 + 2) = 0$

But $3x_1 + 3x_2 + 2 \neq 0$

$\because x_1, x_2 \in \mathbf{N}$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one one

ONTO :

As S = R(f)

& $f : \mathbf{N} \to S$

$\therefore$ C(f) = R(f)

$\therefore f$ is onto

$\therefore f$ is invertible $\Rightarrow \exists f^{-1} : S \to \mathbf{N}$

Let $y = 9x^2 + 6x - 5$

$y = (3x)^2 + 2.3x.1 + (1)^2 - (1)^2 - 5$

$y = (3x + 1)^2 - 6$

$\Rightarrow y + 6 = (3x + 1)^2$

$\Rightarrow 3x + 1 = +\sqrt{y+6}$

$(\because x \in \mathbf{N})$

$\therefore 3x = \sqrt{y+6} - 1$

$$x = \frac{\sqrt{y+6} - 1}{3}$$

$$\therefore f^{-1}(x) = \frac{\sqrt{x+6} - 1}{3}$$

Q10.

Commutative :

let (a, b), (c, d) $\in$ Q $\times$ Q = A

$\Rightarrow$ (a, b) * (c, d) = (ac, b + ad)

& (c, d) * (a, b) = (ca, d + bc)

As (a, b) * (c, d) $\neq$ (c, d) * (a, b)

$\therefore$ * is not commutative

Associative :

Let (a, b), (c, d), (e, f) $\in$ Q $\times$ Q

Consider

((a, b) * (c, d)) * (e, f) = (ac, b + ad) * (e, f) = (ace, b + ad + acf) = (a, b) * ((c, d) * (e, f))

$\therefore$ * is associative.

Identity Element

Let (e, f) be the identity element in A w.r.t. *

$\Rightarrow$ (a, b) * (e, f) = (a, b) $\forall$ (a, b) $\in$ A = Q $\times$ Q

$\Rightarrow$ (ae, b + af) = (a, b)

$\Rightarrow$ ae = a & b + af = b

$\Rightarrow$ e = 1 & f = 0

$\therefore$ (1, 0) $\in$ Q $\times$ Q = A

is the identity element of A w.r.t. *

Inverse Element

let (c, d) $\in$ Q $\times$ Q = A be the inverse element of (a, b) $\in$ Q $\times$ Q

$\Rightarrow$ (a, b) * (c, d) = (1, 0)

$\Rightarrow$ (ac, b + ad) = (1, 0)

$\Rightarrow$ ac = 1 & b + ad = 0

$\Rightarrow$ c = 1/a & d = –b/a ; a $\neq$ 0

$\therefore$ for (a, b) $\in$ Q $\times$ Q; a $\neq$ 0, inverse of (a, b) exists & is given by (1/a, –b/a) $\in$ Q $\times$ Q.

CBSE Exam Pattern Exercise
Objective Questions (2)

1. Which of the following functions are equal
   (a) $\sin^{-1}(\sin x)$ and $\sin(\sin^{-1}x)$
   (b) $\frac{x^2-4}{x-2}, x+2$
   (c) $\frac{x^2}{x}, x$
   (d) A={1,2}, B={3,6}
   f:A→B given by $f(x) = x^2+2$ and
   g:A→B given by g(x) = 3x

2. Let $f : \left[\frac{1}{2}, \infty\right] \to \left[\frac{3}{4}, \infty\right]$, where $f(x) = x^2 - x + 1$ is
   (a) one–one onto
   (b) many one-into
   (c) many one-onto
   (d) one-one into

3. IF A = {1,2,3}
   B = {4,5,6,7} and
   f = {(1,4) (2,5) (3,6)} is a function from A to B then f is
   (a) one-one
   (b) onto
   (c) many one
   (d) both (a) and (b)

4. The range of the function $f(x) = \frac{|x-2|}{x-2}, x \neq 2$ is
   (a) {1,0,–1}
   (b) {1}
   (c) {1,–1}
   (d) None of these

5. f:R→R and g:R→R are given by f(x) = |x| and g(x) = |5x–2|, then fog is
   (a) |5x–2|
   (b) 5x–2
   (c) 2–5x
   (d) None of these

6. Range of the function $f(x) = \frac{|x^2+1|}{x^2+1}$ is
   (a) {1}
   (b) {1,–1}
   (c) {1,0,–1}
   (d) R

7. If set A has 5 elements and set B has three elements then total no. of one-one functions from A to B are
   (a) 0
   (b) $^5P_3$
   (c) 5
   (d) 5!

8. If f(x) = [x] and g(x) = |x| then fog $\left(\frac{-5}{2}\right)$ is
   (where [.] represents greatest integer function of x)
   (a) 2
   (b) 3
   (c) –3
   (d) –2

Answer & Solution

1. (d)

(a) $\sin^{-1}(\sin x) \neq \sin(\sin^{-1}x)$

$\because$ Those functions are equal

whose range f domain are equal

But $\sin^{-1}(\sin x) = x \Rightarrow x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

But $\sin(\sin^{-1}x) = x \Rightarrow x \in [-1,1]$

(b) $\frac{(x^2-4)}{(x-2)} = x+2\ ,\ x \neq 2$

Domain of $\frac{x^2-4}{x-2} \Rightarrow R-2$

& Domain of x+2 is R

Hence they are not equal

(c) same explanation as (b)

(d) A = {1,2}, B = {3,6}

f(1) = 3

f(2) = 6

g(x) = 3x

g(1) = 3

g(2) = 6

Since range & domain in both functions is equal.

Hence functions are equal function

2. (a)

$f(x) = x^2 - x + 1$

$f:\left(\frac{1}{2},\infty\right) \rightarrow \left(\frac{3}{4},\infty\right)$

For one-one

$f(x_1) = f(x_2)$

$x^2_1 - x_1 + 1 = x^2_\alpha - x_1 + 1$

$x^2_1 - x^2_\alpha - x_1 + x_2 = 0$

$(x_1 - x_2)(x_1 + x_2) - 1\ (x_1 - x_2) = 0$

$(x_1 - x_2)(x_1 + x_\alpha - 1) = 0$

either $x_1 = x_\alpha$ or $x_1 + x_\alpha = 1$

But $x_1 + x_\alpha = 1$
only when $x_1 = x_\alpha$
& for no other value
$\therefore\ x_1 = x_2$
Hence one-one
onto
$f(x) = x^2 - x + 1$

$= x^2 - x + \frac{1}{4} - \frac{1}{4} + 1$

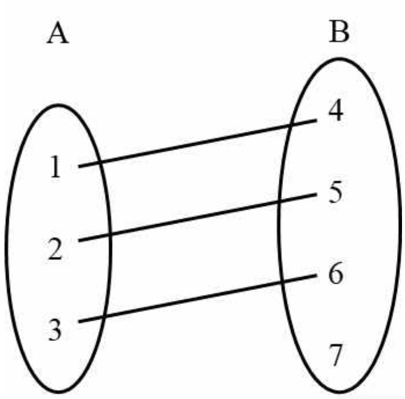
$= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ [Using completing the square method]

For $x \geq \frac{1}{2}$

$y \geq \frac{3}{4}$

Hence range = Codomain
Hence function is onto

3. (a)



Clearly f is one but not onto

4. (c)

$f(x) = \frac{|x-2|}{x-\alpha}, x \neq 2$ is

$f(x) = \begin{cases} 1 & x-2>0 \\ -1 & x-2<0 \end{cases}$

$\therefore$ Range is $\{1, -1\}$

5. (a)
$f(x) = |x|$
$g(x) = |5x - 2|$
$f(g(x)) = ||5x - 2||$
$= |5x - 2|$

6. (a)

$f(x) = \frac{|x^2+1|}{x^2+1}$

x^2+1 is always +ve

$\therefore$ f(x) = 1

{1}

7. (a)

Set A has 5 elements

Set B has 3 elements

$\therefore$ one-one function = zero

8. (a)

f(x) = [x]

g(x) = |x|

f(g(x) = [|x|]

$f\left(g\left(\frac{-5}{2}\right)\right) = \left[\left|\frac{-5}{2}\right|\right] = \left[\frac{5}{2}\right] = 2$