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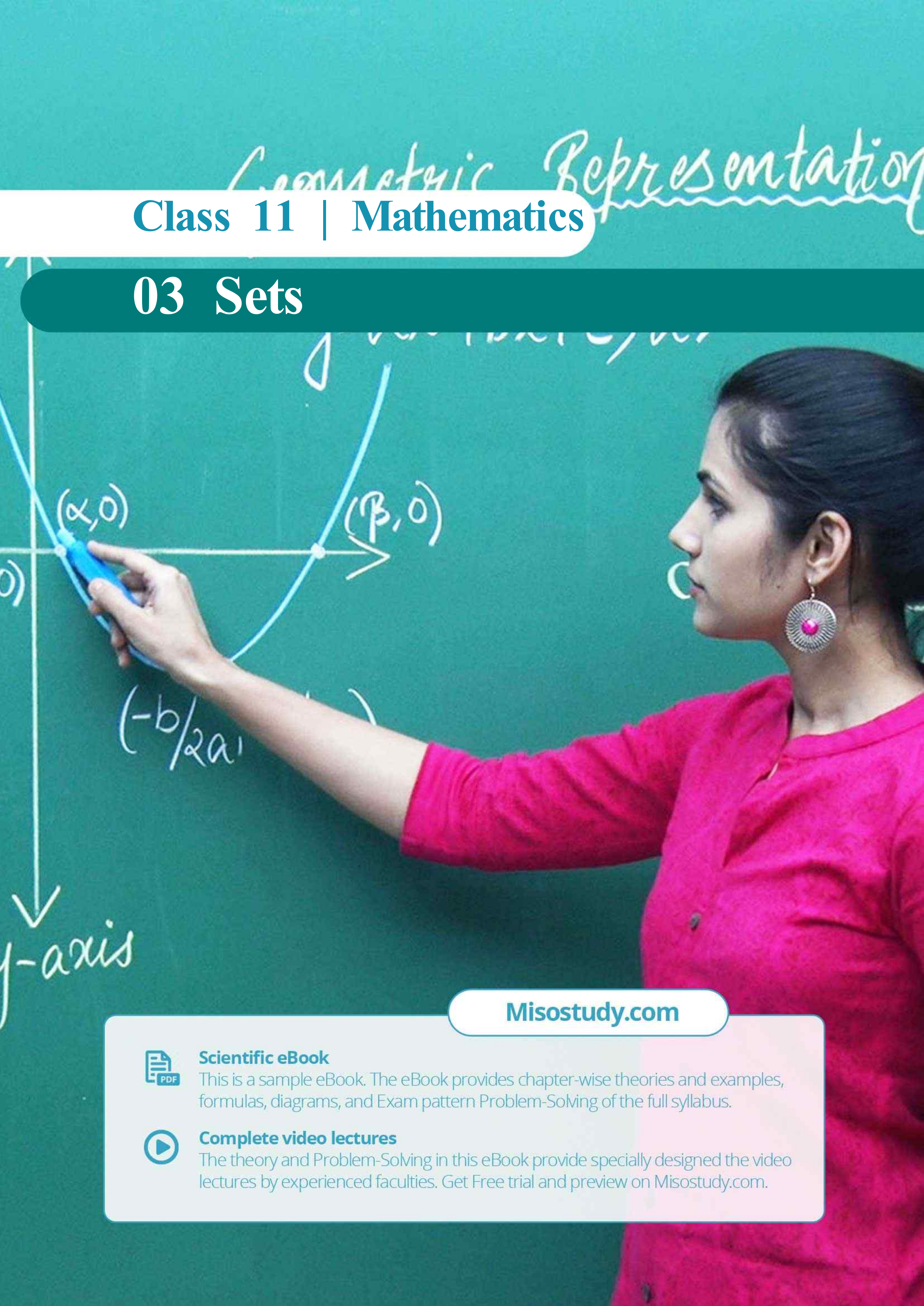


Class 11 | Mathematics

03 Sets

01. Definition of Set

Set as "a well defined collection of objects".

Example The collection of vowels in English alphabets. This set contains five elements, namely, a, e, i, o, u.

NOTE ☞ Collection of good teachers in a school is not a set.

02. Reserved Symbols

We reserve some symbols for these set:

① **N** : for the set of natural numbers.
② **Z** : for the set of integers.
③ $\mathbf{Z}^+$: for the set of all positive integers.
④ **Q** : for the set of all rational numbers.
⑤ $\mathbf{Q}^+$: for the set of all positive rational numbers.
⑥ **R** : for the set of all real numbers.
⑦ $\mathbf{R}^+$: for the set of all positive real numbers.
⑧ **C** : for the set of all complex numbers.

03. Description of a Set

A set is often described in the following two forms. One can make use of any one of these two ways according to his (her) convenience.

(i) Roster form or Tabular form
(ii) Set-builder form

ROSTER FORM

In this form a set is described by listing elements, separated by commas, within braces $\{\}$.

NOTE ☞ (1) The order in which the elements are written in a set makes no difference.
(2) Also, the repetition of an element has no effect.

SET-BUILDER FORM

In this form, a set is described by a characterizing property $P(x)$ of its elements x. In such a case the set is described by $\{x : P(x) \text{holds}\}$ or, $\{x | P(x) \text{holds}\}$, which is read as 'the set of all x such that $P(x)$ holds'. The symbol $'|'$ or $':'$ is read as 'such that'.

04. Types of Sets

(1) EMPTY SET A set is said to be empty or null or void set if it has no element and it is denoted by ϕ(phi).
In Roster method, ϕ is denoted by $\{\}$.
(2) SINGLETON SET A set consisting of a single element is called a singleton set.
(3) FINITE SET A set is called a finite set if it is either void set or its elements can be listed (counted, labelled) by natural numbers $1, 2, 3,$ ... and the process of listing terminates at a certain natural number n (say).
(4) CARDINAL NUMBER OF A FINITE SET The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by $n(A)$.
(5) INFINITE SET A set whose elements cannot be listed by the natural numbers $1, 2, 3, ..., n,$ for any natural number n is called an infinite set.
(6) EQUIVALENT SETS Two finite sets A and B are equivalent if their cardinal numbers are some i.e. $n(A) = n(B)$.
(7) EQUAL SETS Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.

05. Subsets

Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.
If A is a subset of B, we write $A \subseteq B$, which is read as "A is a subset of B" or "A is contained in B".
Thus, $A \subseteq B$ if $a \in A \Rightarrow a \in B$.

SOME RESULTS ON SUBSETS
RESULT 1 Every set is a subset of itself.
RESULT 2 The empty set is a subset of every set.
RESULT 3 The total number of subsets of a finite set containing n elements is 2^n.

REMARK SUBSETS OF THE SET **R** OF REAL NUMBERS

i) The set of all natural numbers $N = \{1, 2, 3, 4, 5, 6, ...\}$
ii) The set of all integers $Z = \{...-3, -2, -1, 0, 1, 2, 3, ...\}$
iii) The set of all rational numbers $Q = \{x : x = \frac{m}{n}, m, n \in Z, n \neq 0\}$.
iv) The set of all irrational numbers. It is denoted by T.

Thus,
$T = \{x : x \in R \text{ and } x \notin Q\}$
Clearly, $N \subset Z \subset Q \subset R$, $T \subset R$ and $N \not\subset T$.

06. Universal Set

A set that contains all sets in a given context is called the universal set.

Example If $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$ and $C = \{1, 3, 5, 7\}$, then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.

07. Power Set

Let A be a set. Then the collection or family of all subsets of A is called the power set of A and is denoted by P(A). The power set of a given set is always non-empty.

Example Let $A = \{1, 2, 3\}$. Then, the subsets of A are:

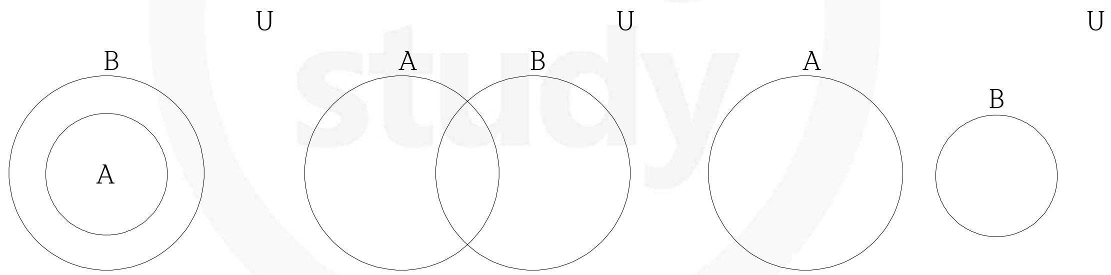
$\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$, and $\{1, 2, 3\}$.

Hence, $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

08. Venn Diagrams

In Venn-diagrams the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle.

A set containing other set **Intersecting sets** **Non intersecting sets**



09. Operations On Sets

(1) **UNION OF SETS**

Let A and B be two sets. The union of A and B is the set of all those elements which belong either to A or to B or to both A and B.

We shall use the notation $A \cup B$ (read as "A union B") to denote the union of A and B.

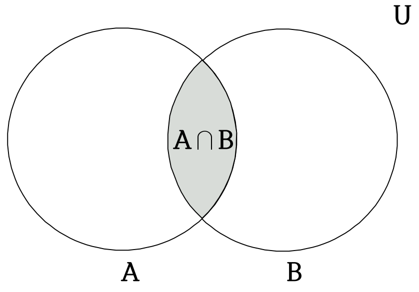
Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

(2) **INTERSECTION OF SETS**

Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.

The intersection of A and B is denoted by $A \cap B$ (read as "A intersection B")

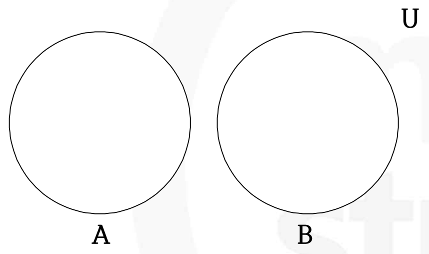
Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.



(3) **DISJOINT SETS**

Two sets A and B are said to be disjoint, if $A \cap B = \phi$.

If $A \cap B \neq \phi$, then A and B said to be intersecting or overlapping sets.

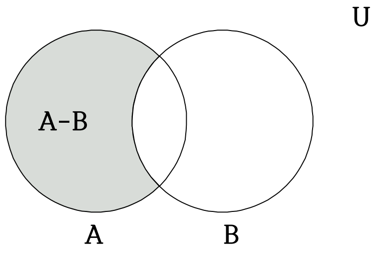


(4) **DIFFERENCE OF SETS**

Let A and B be two sets. The difference of A and B, written as $A - B$, is the set of all those elements of A which do not belong to B.

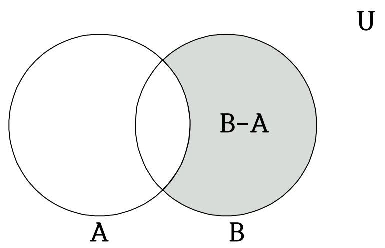
Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

or, $A - B = \{x \in A : x \notin B\}$



Similarly, the difference $B-A$ is the set of all those elements of B that do not belong to A i.e.

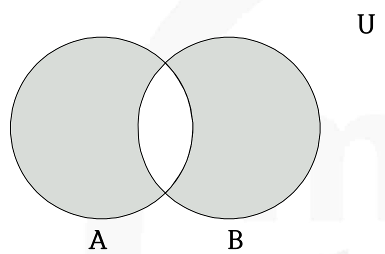
$B-A=\{x\in B: x\notin A\}$.



(5) **SYMMETRIC DIFFERENCE OF TWO SETS**

Let A and B be two sets. The symmetric difference of sets A and B is the set $(A-B)\cup(B-A)$ and is denoted by $A\triangle B$.

Thus, $A\triangle B=(A-B)\cup(B-A)=\{x: x\notin A\cap B\}$

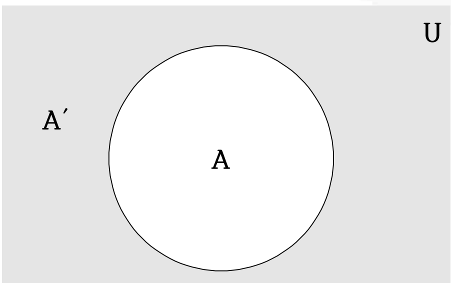


(6) **COMPLEMENT OF A SET**

Let U be the universal set and let A be a set such that $A\subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $U-A$ and is defined the set of all those elements of U which are not in A.

Thus $\quad A'=\{x\in U: x\notin A\}$.

Clearly, $\quad x\in A' \Leftrightarrow x\notin A$.



10. Laws of Algebra of Sets

RESULT 1 (Idempotent Laws) For any set A, we have
(i) $A \cup A = A$

PROOF
$A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$
$A \cap A = A.$
$A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A.$

RESULT 2 (identity Laws) For any set A, we have
(i) $A \cup \phi = A$
i.e. ϕ and U are identity elements for union and intersection respectively.

PROOF
$A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$
(ii) $A \cap U = A.$
$A \cap U = \{x : x \in \text{ and } x \in U\} = \{x : x \in A\} = A$

RESULT 3 (Commutative Laws) For any two sets A and B, we have
(i) $A \cup B = B \cup A$
i.e. union and intersection are commutative.

PROOF
Recall that two sets X and Y are equal iff $X \subseteq Y$ and $Y \subseteq X$. Also, $X \subseteq Y$ if every element of X belongs to Y.
Let x be an arbitrary element of $A \cup B$. Then,
$x \in A \cup B \Rightarrow x \in A$ or $x \in B \Rightarrow x \in B$ or $x \in A \Rightarrow x \in B \cup A$
$\therefore$ $A \cup B \subseteq B \cup A.$
Similarly, $B \cup A \subseteq A \cup B.$
Hence, $A \cup B = B \cup A.$
(ii) $A \cap B = B \cap A$
Let x be an arbitrary element of $A \cap B$.
Then, $x \in A \cap B \Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in B$ and $x \in A \Rightarrow x \in B \cap A$
$\therefore$ $A \cap B \subseteq B \cap A$
Similarly, $B \cap A \subseteq A \cap B$
Hence, $A \cap B = B \cap A.$
RESULT 4 (Associative Laws) If A, B and C are any three sets, then
(i) $(A \cup B) \cup C = A \cup (B \cup C)$
i.e. union and intersection are associative.

PROOF

Let x be an arbitrary element of $(A \cup B) \cup C$. Then,

$x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\therefore$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Hence, $(A \cup B) \cup C = A \cup (B \cup C)$.

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

Let x be an arbitrary element of $A \cup (B \cap C)$. Then,

$x \in A \cap (B \cap C)$

$\Rightarrow$ $x \in A$ and $x \in (B \cap C)$

$\Rightarrow$ $x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow$ $x \in (A \cap B)$ and $x \in C$

$\Rightarrow$ $x \in (A \cap B) \cap C$

$\therefore$ $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

Similarly, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

Hence, $A \cap (B \cap C) = (A \cap B) \cap C$.

RESULT 5 (Distributive Laws) If A, B and C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

i.e. union and intersection are distributive over intersection and union respectively.

PROOF

Let x be an arbitrary element of $A \cup (B \cap C)$. Then,

$x \in A \cup (B \cap C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cap C)$

$\Rightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$ [$\because$ 'or' is distributive over 'and']

$\Rightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow$ $x \in ((A \cup B) \cap (A \cup C))$

$\therefore$ $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Similarly, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) $A \cup (B \cup C) = (A \cap B) \cup (A \cap C)$.

Let x be an arbitrary element of $A \cap (B \cup C)$. Then,

$$
\begin{array}{ll}
 & x \in A \cap (B \cup C) \\
\Rightarrow & x \in A \text{ and } x \in (B \cup C) \\
\Rightarrow & x \in A \text{ and } (x \in B \text{ or } x \in C) \\
\Rightarrow & (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
\Rightarrow & x \in (A \cap B) \text{ or } x \in (A \cap C) \\
\Rightarrow & x \in (A \cap B) \cup (A \cap C) \\
\therefore & A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)
\end{array}
$$

Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

RESULT 6 (De-morgan's Laws) If A and B are any two sets, then

(i) $(A \cup B)' = A' \cap B'$

<u>PROOF</u>

Let x be an arbitrary element of $(A \cup B)'$. Then,

$$
\begin{array}{ll}
 & x \in (A \cup B)' \\
\Rightarrow & x \notin (A \cup B) \\
\Rightarrow & x \notin A \text{ and } x \notin B \\
\Rightarrow & x \in A' \text{ and } x \in B' \\
\Rightarrow & x \in A' \cap B'. \\
\therefore & (A \cup B)' \subseteq A' \cap B'.
\end{array}
$$

Again, let y be an arbitrary element of $A' \cap B'$. Then,

$$
\begin{array}{ll}
 & y \in A' \cap B' \\
\Rightarrow & y \in A' \text{ and } y \in B' \\
\Rightarrow & y \notin A \text{ and } y \notin B \\
\Rightarrow & y \notin A \cup B. \\
\Rightarrow & y \in (A \cup B)' \\
\therefore & A' \cap B' \subseteq (A \cup B)'.
\end{array}
$$

Hence, $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$.

Let x be an arbitrary element of $(A \cap B)'$. Then,

$$
\begin{array}{ll}
 & x \in (A \cap B)' \\
\Rightarrow & x \notin (A \cap B) \\
\Rightarrow & x \notin A \text{ or } x \notin B \\
\Rightarrow & x \in A' \text{ or } x \in B' \\
\Rightarrow & x \in A' \cup B' \\
\Rightarrow & (A \cap B)' \subseteq A' \cup B'.
\end{array}
$$

Again, let y be an arbitrary element of $A' \cup B'$. Then,

$y \in (A' \cap B')$

$\Rightarrow$ $y \in A'$ or $y \in B'$

$\Rightarrow$ $y \notin A$ or $y \notin B$

$\Rightarrow$ $y \notin (A \cap B)$

$\Rightarrow$ $y \in (A \cap B)'$

$\therefore$ $A' \cup B' \subseteq (A \cap B)'$.

Hence, $(A \cap B)' = A' \cup B'$.

11. MORE RESULTS ON OPERATIONS ON SETS

RESULT 1 If A and B are any two sets, then

(i) $A - B = A \cap B'$

PROOF

Let x be an arbitrary element of $A - B$. Then,

$x \in (A - B)$

$\Rightarrow$ $x \in A$ and $x \notin B$

$\Rightarrow$ $x \in A$ and $x \notin B'$

$\Rightarrow$ $x \in A \cap B'$

$\therefore$ $A - B \subseteq A \cap B'$ ...(i)

Again, let y be an arbitrary element of $A \cap B'$. Then,

$y \in A \cap B'$

$\Rightarrow$ $y \in A$ and $y \in B'$

$\Rightarrow$ $y \in A$ and $y \notin B$

$\Rightarrow$ $y \in A - B$

$\therefore$ $A \cap B' \subseteq (A - B)$ ...(ii)

Hence, from (i) and (ii), we have $A - B = A \cap B'$.

(ii) $B - A = B \cap A'$

Let x be an arbitrary element of $B - A$. Then,

$x \in B - A$

$\Rightarrow$ $x \in B$ and $x \notin A$

$\Rightarrow$ $x \in B$ and $x \in A^c$

$\Rightarrow$ $x \in B \cap A^c$

$\therefore$ $B - A \subseteq B \cap A^c$ ...(i)

Again, let y be an arbitrary element of $B \cap A^c$

$\Rightarrow$ $y \in B$ and $y \in A^c$

$\Rightarrow$ $y \in B$ and $y \notin A$

$\Rightarrow$ $y \in B - A$

$\therefore$ $A \cap B^c \subseteq (A - B)$ ...(ii)

Hence, from (i) and (ii), we have $A - B = A \cap B^c$

(iii) $A-B=A \Leftrightarrow A\cap B=\phi$

In order to prove that $A-B=A\Leftrightarrow A\cap B=\phi$

we shall prove that:

(i) $A-B=A\Rightarrow A\cap B=\phi$,

(ii) $A\cap B=\phi \Rightarrow A-B=A$.

First, let $A-B=A$. Then we have to prove that $A\cap B=\phi$. If possible, let $A\cap B\neq\phi$. Then,

$A\cap B\neq\phi \Rightarrow$ there exists $x\in A\cap B$

$\Rightarrow x\in A$ and $x\in B \Rightarrow x\in A-B$ and $x\in B$ $\quad [\because A-B=A]$

$\Rightarrow (x\in A$ and $x\notin B)$ and $x\in B$ $\quad$ [By def. of $A-B$]

$\Rightarrow x\in A$ and $(x\notin B$ and $x\in B)$

But $x\notin B$ and $x\in B$ both can never be possible simultaneously. Thus, we arriver at a contradiction, So, our supposition is wrong.

$\therefore \quad A\cap B=\phi$

Hence, $\quad A-B=A\Rightarrow A\cap B=\phi$ $\quad$ ...(i)

Conversely, let $A\cap B=\phi$. Then we have to prove that $A-B=A$. For this we shall show that $A-B\subseteq A$ and $A\subseteq A-B$

Let x be an arbitrary element of $A-B$. Then,

$x\in A-B \Rightarrow x\in A$ and $x\notin B$

$\Rightarrow x\in A$

$\therefore \quad A-B\subseteq A$.

Again let y be an arbitrary element of A. Then,

$y\in A \Rightarrow y\in A$ and $y\notin B$ $\quad [\because A\cap B=\phi]$

$\Rightarrow y\in A-B$ $\quad$ [By def. of $A-B$]

$\therefore \quad A\subseteq A-B$.

So, we have $A-B\subseteq A$ and $A\subseteq A-B$. Therefore, $A-B=A$.

Thus, $A\cap B=\phi \Rightarrow A-B=A$ $\quad$ ...(i)

Hence, from (i) and (ii), we have

$A-B=A \Leftrightarrow A\cap B=\phi$.

(iv) $(A-B)\cup B=A\cup B$

Let x be an arbitrary element of $(A-B)\cup B$. Then,

$x\in (A-B)\cup B$

$\Rightarrow \quad x\in A-B$ or $x\in B$

$\Rightarrow \quad (x\in A$ and $x\notin B)$ or $x\in B$

$\Rightarrow \quad (x\in A$ or $x\in B)$ and $(x\notin B$ or $x\in B)$

$\Rightarrow \quad x\in A\cup B$

$\therefore \quad (A-B)\cup B\subseteq A\cup B$

Let y be an arbitrary element of $A\cup B$. Then,

$y\in A\cup B$

$\Rightarrow \quad y\in A$ or $y\in B$

$\Rightarrow \quad (y\in A$ or $y\in B)$ and $(y\notin B$ or $y\in B)$

$\Rightarrow \quad (y\in A$ and $y\notin B)$ or $y\in B$

$\Rightarrow \quad y\in (A-B)\cup B$

$\therefore \quad A\cup B\subseteq (A-B)\cup B$

Hence, $\quad (A-B)\cup B=A\cup B$.

(v) $(A-B)\cap B=\phi$

If possible let $(A-B)\cap B\neq\phi$. Then, there exists at least one element x, (say), in $(A-B)\cap B$.

Now, $x\in(A-B)\cap B \Rightarrow x\in(A-B)$ and $x\in B$

$\Rightarrow (x\in A \text{ and } x\notin B)$ and $x\in B$

$\Rightarrow x\in A$ and $(x\notin B \text{ and } x\in B)$

But, $x\notin B$ and $x\in B$ both can never be possible simultaneously. Thus, we arrive at a contradiction. So, our supposition is wrong.

Hence, $(A-B)\cap B=\phi$

(vi) $A\subseteq B\Leftrightarrow B'\subseteq A'$

First, let $A\subseteq B$. Then we have to prove that $B'\subseteq A'$. Let x be an arbitrary element of B'. Then,

$x\in B' \Rightarrow x\notin B$

$\Rightarrow x\notin A$ $\quad [\because A\subseteq B]$

$\Rightarrow x\in A'$

$\therefore$ $B'\subseteq A'$.

Thus, $A\subseteq B \Rightarrow B'\subseteq A'$ …(i)

Conversely, let $B'\subseteq A'$. Then, we have to prove that $A\subseteq B$. Let y be an arbitrary element of A. Then,

$y\in A \Rightarrow y\notin A'$

$\Rightarrow y\notin B'$ $\quad [\because B'\subseteq A']$.

$\Rightarrow y\in B$

$\therefore$ $A\subseteq B$.

Thus, $B'\subseteq A' \Rightarrow A\subseteq B$ …(ii)

From (i) and (ii), we have $A\subseteq B \Leftrightarrow B'\subseteq A'$.

(vii) $(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$

Let x be an arbitrary element of $(A-B)\cup(B-A)$. Then,

$x\in(A-B)\cup(B-A)$

$\Rightarrow$ $x\in A-B$ or $x\in B-A$

$\Rightarrow$ $(x\in A \text{ and } x\notin B)$ or $(x\in B \text{ and } (x\notin A)$

$\Rightarrow$ $(x\in A \text{ or } x\in B)$ and $(x\notin B \text{ or } x\notin A)$

$\Rightarrow$ $x\in(A\cup B)$ and $x\notin(A\cap B)$

$\Rightarrow$ $x\in(A\cup B)-(A\cap B)$

$\therefore$ $(A-B)\cup(B-A)\subseteq(A\cup B)-(A\cap B)$ …(i)

Again, let y be an arbitrary element of $(A\cup B)-(A\cap B)$.

Thus, $y\in(A\cup B)-(A\cap B)$

$\Rightarrow$ $y\in A\cup B$ and $y\notin A\cap B$

$\Rightarrow$ $(y\in A \text{ or } y\in B)$ and $(y\notin A \text{ and } y\notin B)$

$\Rightarrow$ $(y\in A \text{ and } y\notin B)$ or $(y\in B \text{ and } y\notin A)$

$\Rightarrow$ $y\in(A-B)$ or $y\in(B-A) \Rightarrow y\in(A-B)\cup(B-A)$.

$\therefore$ $(A\cup B)-(A\cap B)\subseteq(A-B)\cup(B-A)$ …(ii)

Hence, from (i) and (ii), we have

$(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$.

RESULT 2 If A, B and C are any three sets, then prove that:
(i) $A-(B\cap C)=(A-B)\cup(A-C)$

PROOF
Let x be any element of $A-(B\cap C)$. Then,

$$\begin{aligned} x\in A-(B\cap C) &\Rightarrow x\in A \text{ and } x\not\in(B\cap C)\\ &\Rightarrow x\in A \text{ and } (x\not\in B \text{ or } x\not\in C)\\ &\Rightarrow (x\in A \text{ and } x\not\in B) \text{ or } (x\in A \text{ and } x\not\in C)\\ &\Rightarrow x\in(A-B) \text{ or } x\in(A-C)\\ &\Rightarrow x\in(A-B)\cup(A-C)\end{aligned}$$

$\therefore$ $A-(B\cap C)\subseteq(A-B)\cup(A-C)$
Similarly, $(A-B)\cup(A-C)\subseteq A-(B\cap C)$
Hence, $A-(B\cap C)=(A-B)\cup(A-C)$

(ii) $A-(B\cup C)=(A-B)\cap(A-C)$
Let x be an arbitrary element of $A-(B\cup C)$. Then

$$\begin{aligned} x\in A-(B\cup C) &\Rightarrow x\in A \text{ and } x\not\in(B\cap C)\\ &\Rightarrow x\in A \text{ and } (x\not\in B \text{ and } x\not\in C)\\ &\Rightarrow (x\in A \text{ and } x\not\in B) \text{ and } (x\in A \text{ and } x\not\in C)\\ &\Rightarrow x\in(A-B) \text{ and } x\in A-C\\ &\Rightarrow x\in(A-B)\cap(A-C)\end{aligned}$$

$\therefore$ $A-(B\cup C)\subseteq(A-B)\cap(A-C)$
Similarly, $(A-B)\cap(A-C)\subseteq A-(B\cup C)$
Hence, $A-(B\cup C)=(A-B)\cap(A-C)$

(iii) $A\cap(B-C)=(A\cap B)-(A\cap C)$
Let x be any arbitrary element of $A\cap(B-C)$. Then

$$\begin{aligned} x\in A\cap(B-C) &\Rightarrow x\in A \text{ and } x\in(B-C)\\ &\Rightarrow x\in A \text{ and } (x\in B \text{ and } x\not\in C)\\ &\Rightarrow (x\in A \text{ and } x\in B) \text{ and } (x\in A \text{ and } x\not\in C)\\ &\Rightarrow x\in(A\cap B) \text{ and } x\not\in(A\cap C)\\ &\Rightarrow x\in(A\cap B)-(A\cap C)\end{aligned}$$

$\therefore$ $A\cap(B-C)\subseteq(A\cap B)-(A\cap C)$
Similarly, $(A\cap B)-(A\cap C)\subseteq A\cap(B-C)$
Hence, $A\cap(B-C)=(A\cap B)-(A\cap C)$.

(iv) $A\cap(B\triangle C)=(A\cap B)\triangle(A\cap C)$

$$\begin{aligned} A\cap(B\triangle C) &= A\cap[(B-C)\cup(C-B)]\\ &=[A\cap(B-C)]\cup[A\cap(C-B)] && \text{[By distributive law]}\\ &=[(A\cap B)-(A\cap C)]\cup[(A\cap C)-(A\cap B)] && \text{[Using (iii)]}\\ &=(A\cap B)\triangle(A\cap C)\end{aligned}$$

12. SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \triangle B) =$ No. of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ $\quad [\because (A - B)$ and $(B - A)$ are disjoint$]$

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$
$- n(A \cap C) + n(A \cap B \cap C)$

(vi) No. of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3\,n(A \cap B \cap C)$.

(vii) No. of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C)$
$- 2n(A \cap C) + 3n(A \cap B \cap C)$

(viii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(ix) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$.

JEE Main Pattern
Exercise (1)

Q1. For two events A and B which of the following is simple expression of $(A\cap B)\cup(A\cap B')\cup(A'\cap B)$?
(a) $(A\cap B)$
(b) $(A\cup B)$
(c) $(A'\cap B')$
(d) $(A\cap B')$

Q2. If $U=\{1,2,3\}$ and $A=\{1,2\}$ then
$[P(A)]'=$ __________
(a) $\{\{3\},\{2,3\},\{1,3\},\{1,2\},\phi\}$
(b) $\{\{3\},\{2,3\},\{1,3\},\{1,2,3\}\}$
(c) $\{\{3\},\{2,3\},\{1,3\},\{1,2,3\},\phi\}$
(d) $\{\{3\},\{2,3\},\{1,3\},\{1,2\}\}$

Q3. Let U be the universal set and $A\cup B\cup C=U$. Then $[(A-B)\cup(B-C)\cup(C-A)]'$ equals
(a) $A\cup B\cup C$
(b) $A\cap B\cap C$
(c) $A\cup(B\cap C)$
(d) $A\cap(B\cup C)$

Q4. The set $(A\cup B\cup C)\cap(A\cap B'\cap C')'\cap C'$ equals
(a) $B\cap C'$
(b) $B\cup C'$
(c) $A\cap C$
(d) $A\cup C$

Q5. Let $A=\{(x,y):y=e^{x},x\in R\}$, $B=\{(x,y):y=e^{-x},x\in R\}$, then
(a) $A\cap B=\phi$
(b) $A\cap B\neq\phi$
(c) $A\cup B=R$
(d) $A\cup B=A$

Q6. Let U be the universal set containing 700 elements. If A,B are sub-sets of U such that $n(A)=200, n(B)=300$ and $n(A\cap B)=100$. Then, $n(A'\cap B')=$
(a) 400
(b) 600
(c) 300
(d) none of these

Q7. If $A=\{1,2,3,4,5\}$, then the number of proper subset of A is

(a) 120
(b) 30
(c) 31
(d) 32

Q8. If A and B are two sets then $B-(B-A)=$____

(a) $(A-B)-B$
(b) $A-(A-B)$
(c) A
(d) B

Q9. Taking $U=[1,5]$, $A=\{x/x\in N, x^2-6x+5=0\}$, $A'=$___

(a) $\{1,5\}$
(b) $(1,5)$
(c) $[1,5]$
(d) $[-1,-5]$

Q10. If A and B are two sets such that $n(A)=115, n(B)=326, n(A-B)=47$, then write $n(A\cup B)=$

(a) 372
(b) 373
(c) 400
(d) none of these

Answer & Solution

ANSWER

Q1	Q2	Q3	Q4	Q5
(b)	(b), (c), (d)	(b), (c), (d)	(a)	(b)
Q6	Q7	Q8	Q9	Q10
(c)	(c)	(b)	(b)	(b)

JEE Advanced Pattern
Exercise (1)

Q1. Two finite sets have m and n element respectively. The total number of subsets of first set is 112 more than the total number of subsets of the second set. The value of m and n respectively are:

(a) 5.2
(b) 4.7
(c) 7.4
(d) 2.5

Q2. A survey shows that 70% of the Indians like mango wheres 82% like apple. If x% of Indian like both mango and apples then:

(a) $x=52$
(b) $52 \leq x \leq 70$
(c) $x=70$
(d) $70 \leq x \leq 82$

Q3. If $X \cup \{3,4\} = \{1,2,3,4,5,6\}$ the which of the following is true

(a) Smallest set $X=\{1,2,5,6\}$
(b) Smallest set $X=\{1,2,3,5,6\}$
(c) Smallest set $X=\{1,2,3,4\}$
(d) Greatest set $X=\{1,2,3,4\}$

Q4. In a certain town 30% families own a scooter and 40% on a car 50% own neither a scooter nor a car 2000 families own both a scooter and car consider the following statements in this regard

(1) 20% families own both scooter and car
(2) 35% families own either a car or a scooter
(3) 10000 families live in town.

Which of the above statement are correct ?

(a) 2 and 3
(b) 1, 2 and 3
(c) 1 and 2
(d) 1 and 3

Q5. Let $A=\{\theta : \tan\theta + \sec\theta = \sqrt{2}\sec\theta\}$ and $B=\{\theta : \sec\theta - \tan\theta = \sqrt{2}\tan\theta\}$ be two sets then.

(a) $A=B$
(b) $A \subset B$
(c) $A \neq B$
(d) $B \subset A$

Q6. A d B are two sets $n(A-B)=8+2x$, $n(B-A)=6x$ and $n(A\cap B)=x$. If $n(A)=n(B)$ then $n(A\cap B)=$_____

(a) 26
(b) 50
(c) 24
(d) none of these

Q7. $A=\{(a,b)/b=2a-5\}$ If $(m,5)$ and $(6,n)$ are the member of set A then m and n are respectively

(a) 5,7
(b) 7,5
(c) 2,3
(d) 5,3

Q8. There are three-three sets given in column-A and column-B

Column-A	Column-B
(1) $\{L,A,T\}$	(A) $\{x/x\in z, x^2<5\}$
(2) $\{x\in z/x^3-x=0\}$	(B) $\{x/x$ is a letter of the word LATA$\}$
(3) $\{-2,-1,0,1,2\}$	(C) $\left\{\sin 0, \sin\frac{3\pi}{2}, \tan\frac{5\pi}{4}\right\}$

Which one of the following matches is correct?

(a) $1-A,\ 2-B,\ 3-C$
(b) $1-B,\ 2-A,\ 3-C$
(c) $1-B,\ 2-C,\ 3-A$
(d) $1-A,\ 2-C,\ 3-B$

Q9. If $A=\{n^3+(n+1)^3+(n+z)^3; n\in N\}$ and $B=\{9n, n\in N\}$ then

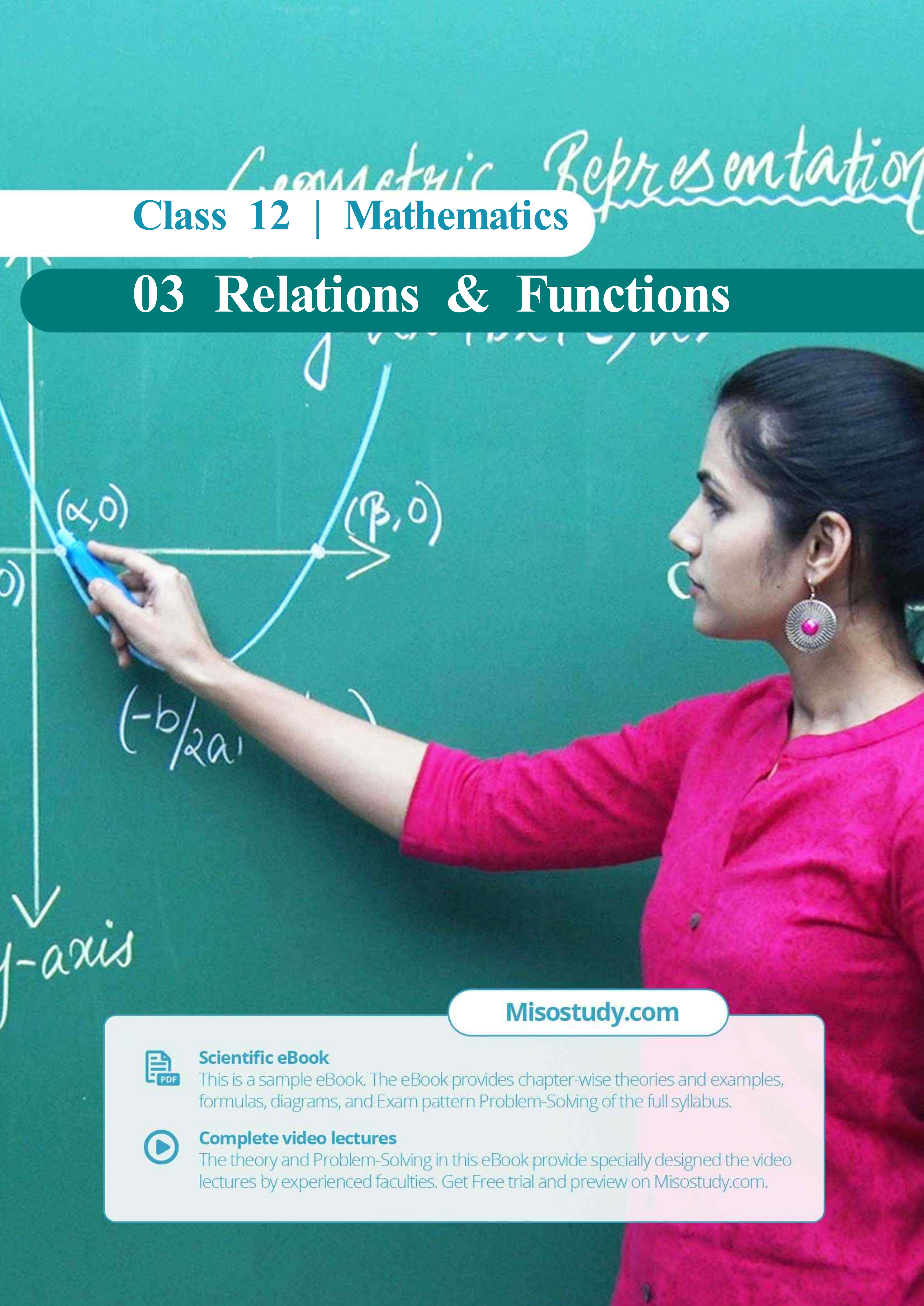
(a) $A\subset B$
(b) $B\subset A$
(c) $A=B$
(d) $A'=B$

Q10. If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A\cup B$? Find also, the maximum number of elements in $A\cup B$.
(a) 9, 6
(b) 8, 5
(c) 7, 4
(d) 6, 3

Answer & Solution

ANSWER

Q1	Q2	Q3	Q4	Q5
(c)	(b)	(a)	(d)	(a)
Q6	Q7	Q8	Q9	Q10
(d)	(a)	(c)	(a)	(a)



Class 12 | Mathematics

03 Relations & Functions

01. Types of Relations

(A) VOID, UNIVERSAL AND IDENTITY RELATIONS

Void Relation-

Let A be a set. Then, $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on set A.

Universal Relation-

Let A be a set. Then, $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

NOTE ☞ It is to note here that the void relation and the universal relation on a set A are respectively the smallest and the largest relations on set A.
Both the empty (or void) relation and the universal relation are sometimes. They are called trivial relations.

Identity Relation-

Let A be a set. Then, the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation i.e., if every element of A is related to itself only.

(B) REFLEXIVE, SYMMETRIC, TRANSITIVE, ANTISYMMETRIC RELATIONS

Reflexive Relation-

A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Symmetric Relation-

A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.

Transitive Relation-

Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $(a, c) \in R$ for all $a, b, c \in A$.

i.e. aRb and bRc

$\Rightarrow$ aRc for all $a, b, c \in A$.

Antisymmetric Relation-

Let A be any set. A relation R on set A is said to be an antisymmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

NOTE ☞ It follows from this definition that if $(a, b) \in R$ and $(b, a) \notin R$, then also R is an antisymmetric relation.

(C) **Equivalence Relation**

A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

02. Some Results on Relations

RESULT 1

If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

OR

The intersection of two equivalence relations on a set is an equivalence relation on the set.

RESULT 2

The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

RESULT 3

If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

OR

The inverse of an equivalence relation is an equivalence relation.

03. Kinds of Functions

ONE-ONE FUNCTION (INJECTION)

A function $f : A \to B$ is said to be a one-one function or an injection if different elements of A have different images in B.

Thus, $f : A \to B$ is one-one

$\Leftrightarrow$ $a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$

$\Leftrightarrow$ $f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$

Algorithm

(i) Take two arbitrary elements *x, y* (say) in the domain *of f.*

(ii) Put *f(x) = f(y)*

(iii) Solve *f(x) = f(y).* If *f(x) = f(y* gives x = y only, them $f : A \to B$ is a one-one function (or an injection). Otherwise not.

NOTE ☞ (i) Let $f : A \to B$ and let $x, y \in A$. Then, $x = y \Rightarrow f(x) = f(y)$ is always true from the definition. But, $f(x) = f(y) \Rightarrow x = y$ is true only when f is one-one.

(ii) If A and B are two sets having m and n elements respectively such that $m \le n$, then total number of one-one functions from A to B is ${}^nC_m \times m!$.

MANY-ONE FUNCTION

A function $f : A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

Thus, $f : A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \ne y$ but *f(x) = f(y)*.

NOTE ☞ In other words, $f : A \to B$ is many-one function if it is not a one-one function.

ONTO FUNCTION (SURJECTION)

A function $f : A \to B$ is said to be an onto function or a surjection if every element of B is the f-image of some element of A i.e., if f(A) = B of range of f is the co-domain of f.

Thus, $f : A \to B$ is a surjection iff for each $b \in B$, there exists $a \in A$ such that f(a) = b.

INTO FUNCTION. A function $f : A \to B$ is an into function if there exists an element in B having no pre-image in A.

In other words, $f : A \to B$ is an into function if it is not an onto function.

Algorithm

Let $f : A \to B$ be the given function.

(i) Choose an arbitrary element y in B.

(ii) Put f(x) = y

(iii) Solve the equation f(x) = y for x and obtain x in terms of y. Let x = g(y).

(iv) If for all values of $y \in B$, the values of x obtained from x = g(y) are in A, then f is onto. If there are some $y \in B$ for which x, given by x = g(y), is not in A. Then, f is not onto.

BIJECTION (ONE-ONE ONTO FUNCTION)

A function $f : A \to B$ is a bijection if it is one-one as well as onto. In other words, a function $f : A \to B$ is a bijection, if

(i) it is one-one i.e. $f(x) = f(y) \Rightarrow x = y$ for all x, $y \in A$.

(ii) it is onto i.e. for all $y \in B$, there exists $x \in A$ such that f(x) = y.

REMARK If A and B are two finite sets and $f : A \to B$ is a function, then

(i) $f \, is \, an \in jection \Rightarrow n(A) \le n(B)$

(ii) $f \, is \, an \, surjection \Rightarrow n(B) \le n(A)$

(iii) $f \, is \, an \, bijection \Rightarrow n(A) = n(B)$.

HOMOGENEOUS FUNCTIONS

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2+3y^2-xy$ is homogeneous in x & y.

i.e. $f(x,y)$ is a homogeneous function iff

$$f(tx,ty)=t^n f(x,y)$$

or $f(x,y)=x^n g\left(\frac{y}{x}\right)=y^n h\left(\frac{x}{y}\right)$, where n is the degree of homogenity

e.g. $f(x,y)=\dfrac{x-y\cos x}{y\sin+x}$ is not a homogeneous function and

e.g. $f(x,y)=\frac{x}{y}\ln\frac{y}{x}+\frac{y}{x}\ln\frac{x}{y};\ \sqrt{x^2-y^2}+x;\ x+y\cos\frac{y}{x}$ are homogeneous functions of degree one.

BOUNDED FUNCTION

A function is said to be bounded if $|f(x)|\le M$, where M is a finite quantity.

e.g. $f(x)=\sin x$ is bounded in $[-1,1]$

IMPLICIT & EXPLICIT FUNCTION

A function defined by an equation not solved for the dependent variable is called an IMPLICIT FUNCTION. For eg. the equation $x^3+y^3=1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an EXPLICIT FUNCTION.

ODD & EVEN FUNCTION

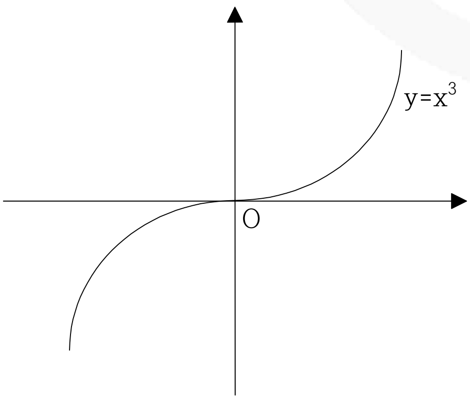
A function $f(x)$ defined on the symmetric interval $(-a,a)$

Iff $f(-x)=f(x)$ for all x in the domain of $'f'$ then f is said to be an even function.

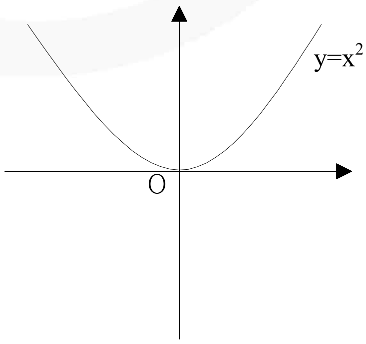
e.g. $f(x)=\cos x; g(x)=x^2+3$

Iff $f(-x)=-f(x)$ for all x in the domain of $'f'$ then f is said to be an odd function.

e.g. $f(x)=\sin x;\ g(x)=x^3+x$.



Odd function (Symmetric about origin)



Even function (Symmetric about y-axis)

NOTE ☞ **(a)** $f(x)-f(-x)=0 => f(x)$ is even & $f(x)+f(-x)=0 => f(x)$ is odd.

(b) A function may neither be odd nor even.

(c) Inverse of an even function is not defined and an even function can not be strictly monotonic

(d) Every even function is symmetric about the $y-$axis & every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even & an odd function.

e.g. $$f(x)=\underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}}+\underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}} \qquad 2^x=\underbrace{\frac{2^x+2^{-x}}{2}}_{\text{EVEN}}+\underbrace{\frac{2^x-2^{-x}}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x)=0$. Any non zero constant is even.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd.

$f(x)$	$g(x)$	$f(x)+g(x)$	$f(x)-g(x)$	$f(x).g(x$	$f(x)/g(x$	$(gof)(x$	$(fog)(x$
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

PERIODIC FUNCTION

A function $f(x)$ is called periodic if there exists a positive number $T(T>0)$ called the period of the function such that $f(x+T)=f(x)$, for all values of x within the domain of x.
e.g. The function $\sin x$ & $\cos x$ both are periodic over 2π & $\tan x$ is periodic over π.

Graphically :

If the graph repeats at fixed interval then function is said to be periodic and its period is the width of that interval.

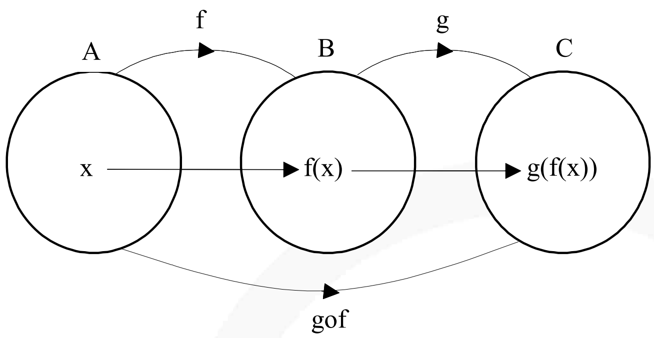
PROPERTIES OF PERIODIC FUNCTION

(i) $f(T)=f(0)=f(-T)$, where $'T'$ is the period.
(ii) Inverse of a periodic function does not exist.
(iii) Every constant function is always periodic, with no fundamental period.
(iv) If $f(x)$ has a period p, then $\dfrac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.
(v) if $f(x)$ has a period T then $f(ax+b)$ has a period $\dfrac{T}{|a|}$.

(vi) If $f(x)$ has a period T & $g(x)$ also has a period T then it does not mean that $f(x)+g(x)$ must have a period T. e.g. $f(x)=|\sin x|+|\cos x|; \sin^4 x+\cos^4 x$ has fundamental period equal to $\frac{\pi}{2}$.

(vii) If $f(x)$ and $g(x)$ are periodic then $f(x)+g(x)$ need not be periodic.
e.g. $f(x)=\cos x$ and $g(x)=\{x\}$

04. Composition of Uniformly Defined Functions



Definition:

Let $f: A \to B$ and $g: B \to C$ be two functions. Then a function $gof: A \to C$ defined by

$(gof)(x) = g(f(x))$, for all $x \in A$

is called the composition of f and g.

NOTE ☞ (i) It is evident from the definition that gof is defined only if for each $x \in A$, f(x) is an element of g so that we can take its g-image. Hence, for the composition gof to exist, the range of f must be subset of the domain of g.

(ii) It should be noted that gof exists iff the range of f is a subset of domain of g. Similarly, fog exists if range of g is a subset of domain of f.

PROPERTIES OF COMPOSITION OF FUNCTIONS

RESULT 1 The composition of functions is not commutative i.e. $fog \neq gof$.

RESULT 2 The composition of functions is associative i.e. *if f, g, h* are three functions such that *(fog)oh* and *fo(goh)* exist, then

$(fog)oh = fo(goh)$

RESULT 3 The composition of two bijections is a bijection i.e. if f and g are two bijections, then gof is also a bijection.

RESULT 4 Let $f: A \to B$. Then, $foI_A = I_B of = f$ i.e. the composition of any function with the identity function is the function itself.

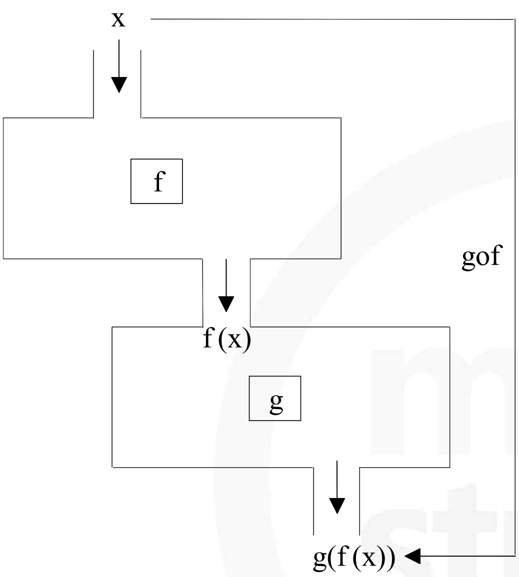
RESULT 5 Let $f: A \to B$, $g: B \to A$ be two functions such that $gof = I_A$. Then, f is an injection and g is a surjection.

RESULT 6 Let $f : A \to B$ and $g : B \to A$ be two function such that $fog = I_B$. Then, f is a surjection and g is an injection.

RESULT 7 Let $f : A \to B$ and $g : B \to A$ be two functions. Then,

(i) $gof : A \to C$ is into $\Rightarrow g : B \to C$ is onto
(ii) $gof : A \to C$ is one-one $\Rightarrow f : A \to B$ is one-one
(iii) $gof : A \to C$ is onto and $\Rightarrow g : B \to C$ is one-one $\Rightarrow f : A \to B$ is onto
(iv) $gof : A \to C$ is one-one and $\Rightarrow f : A \to B$ is onto $\Rightarrow g : B \to C$ is one-one.

05. Composition of Non-Uniformly Defined Functions



Piecewise or non-uniformly defined functions: Those functions whose domain is divided into two or more than two parts so that the function has different analytical formulae in different parts of its domain are called piecewise or non-uniformly defined functions. Also, a piecewise defined function is composed of branches of two or more functions.

Method: The method to find the composition of two non-uniformly defined functions is as follows-

Consider the functions as defined below

$$f(x) = \begin{cases} 2x-1; & 0 \le x < 2 \\ x^2+1; & 2 \le x \le 4 \end{cases} \quad \& \quad g(x) = \begin{cases} x+1; & -1 \le x < 1 \\ 2x; & 1 \le x \le 3 \end{cases}$$

Let us fined the composite function $fog(x)$. The following steps are involved.

STEP 1: Replace $g(x)$ in the place of x in the definition of $f(x)$.

i.e., $f(g(x)) = \begin{cases} 2g(x)-1; & 0 \le g(x) < 2 \\ (g(x))^2+1; & 2 \le g(x) \le 4 \end{cases}$

STEP 2: Apply the definition of $g(x)$ in the above step (1).

$$\Rightarrow f(g(x)) = \begin{cases} 2(x+1)-1; & -1 \le x < 1 \quad \& \quad 0 \le x+1 < 2 \\ 2(2x)-1; & 1 \le x \le 3 \quad \& \quad 0 \le 2x < 2 \\ (x+1)^2+1; & -1 \le x < 1 \quad \& \quad 2 \le x+1 \le 4 \\ (2x)^2+1; & 1 \le x \le 3 \quad \& \quad 2 \le 2x \le 4 \end{cases}$$

STEP 3: Take the intersection of domain and find the final definition.

$$\text{i.e., } f(g(x)) = \begin{cases} 2(x+1)-1; & -1 \le x < 1 \\ 2(2x)-1; & x \in \{\ \} \\ (x+1)^2+1; & x \in \{\ \} \\ (2x)^2+1; & 1 \le x \le 2 \end{cases}$$

$$\Rightarrow f(g(x)) = \begin{cases} 2x+1; & -1 \le x < 1 \\ 4x^2+1; & 1 \le x \le 2 \end{cases}$$

Thus, the domain of composite function $fog(x)$ is $[-1, 2]$

06. Composition of Real Functions

Definition:

Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then,

$$gof : X = \{x \in D_1 : f(x) \in D_2\} \to R$$

and,

$$fog : Y = \{x \in D_2 : g(x) \in D_1\} \to R$$

are defined as

$$gof\,(x) : g(f(x)) \text{ for all } x \in X \text{ and } fog\,(x) = f(g(x)) \text{ for all } x \in Y.$$

REMARK

(1) If Range $(f) \subseteq$ Domain (g), then $gof : D_1 \to R$ and if Range $(g) \subseteq$ Domain (f), then $fog : D_2 \to R$.

(2) For any two real functions f and g, it may be possible that gof exists but fog does not. In some cases, even if both exist, they may not be equal.

(3) If Range $(f) \cap$ Domain $(g) = \phi$, then gof does not exist. In order words, gof exists if Range $(f) \cap$ Domain $(g) \neq \phi$.
Similarly, fog exists if range $(g) \cap$ Domain $(f) \neq \phi$.

(4) If f and g are bijections, then fog and gof both are bijections.

(5) If $f : R \to R$ and $g : R \to R$ to real functions, then fog and gof both exist.

07. Inverse of An Element

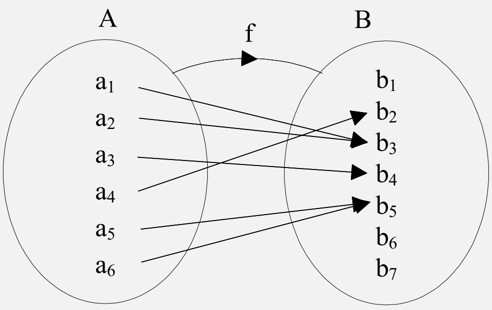
Let A and B be two sets and let $f: A \rightarrow B$ be a mapping. If $a \in A$ is associated to $b \in B$ under the function f, then $'b'$ is called the f image of $'a'$ and we write it as $b = f(a)$. We also say that $'a'$ is the pre-image or inverse element of $'b'$ under f and we write $a = f^{-1}(b)$.

NOTE ☞ The inverse of an element under a function may consist of a single element, two or more elements or no element depending on whether function is injective or many-one; onto or into.

If f is represented by Figure, then we find that

$$f^{-1}(b_1) = \phi, f^{-1}(b_2) = a_{4,}$$
$$f^{-1}(b_3) = \{a_1, a_2\}, f^{-1}(b_4) = a_{3,}$$
$$f^{-1}(b_5) = \{a_5, a_6\}, f^{-1}(b_6) = \phi$$

and, $$f^{-1}(b_7) = \phi$$



08. Inverse of A Function

Definition:

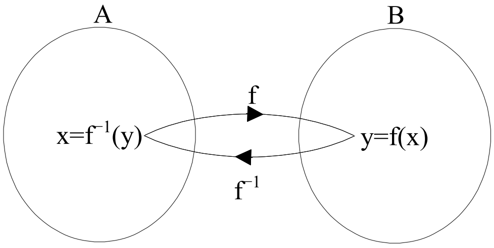
Let $f: A \rightarrow B$ be a bijection. Then a function $g: B \rightarrow A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of f.

i.e., $f(x) = y \Leftrightarrow g(y) = x$

The inverse of f is generally denoted by f^{-1}

Thus, if $f: A \rightarrow B$ is a bijection, then $f^{-1}: B \rightarrow A$ is such that

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$



Algorithm

Let $f: A \to B$ be a bijection. To find the inverse of f we follow the following steps:

STEP I Put $f(x) = y$, where $y \in B$ and $x \in A$.

STEP II Solve $f(x) = y$ to obtain x in terms of y.

STEP III In the relation obtained in step II replace x by $f^{-1}(y)$ to obtain the required inverse of f.

09. Properties of Inverse of a Function

RESULT 1 The inverse of a bijection is unique.

RESULT 2 The inverse of a bijection is also a bijection.

RESULT 3 If $f: A \to B$ is a bijection and $g: B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity function on the sets A and B respectively.

RESULT 4 If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is a bijection and $(gof)^{-1} = f^{-1} og^{-1}$

RESULT 5 If $f: A \to B$ and $g: B \to A$ be two functions such that $gof = I_A$ and $fog = I_B$. Then, f and g are bijections and $g = f^{-1}$.

RESULT 6 Let $f: A \to B$ be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

REMARK

(1) Sometimes $f: A \to B$ is one-one but not onto. In such a case f is not invertible. But, $f: A \to Range(f)$ is both one and onto. So, it is invertible and its inverse can be found.

(2) Result 5 suggests us an alternative method to prove the invertibility of a function. It states that if $f: A \to B$ and $g: B \to A$ are two functions such that $gof = I_A$ and $fog = I_B$, then f and g are inverse of each other.

Result 5 suggests the following algorithm to find the inverse of an invertible function.

Algorithm

STEP I Obtain the function and check its bijectivity.

STEP II If f is a bijection, then it is invertible.

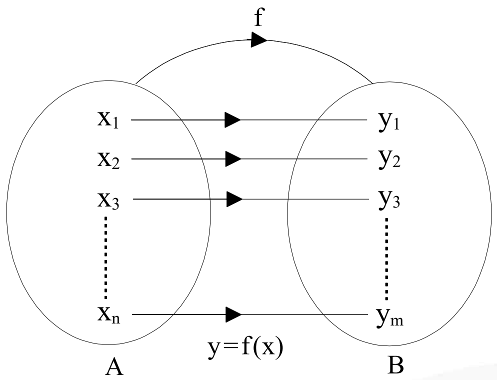
STEP III Use the formula for $f(x)$ and replace x by $f^{-1}(x)$ in it to obtain the LHS of $f(f^{-1}(x)) = x$.

Solve this equation for $f^{-1}(x)$ to get $f^{-1}(x)$.

10. Permutation and Combinations Problems

NUMBER OF RELATIONS AND FUNCTIONS

Given two finite sets A and B having n and m elements respectively, i.e., $n(A)=n$ and $n(B)=m$.



Number of Relations: No. of relations = Number of subsets of $A\times B = 2^{n(A\times B)} = 2^{nm}$

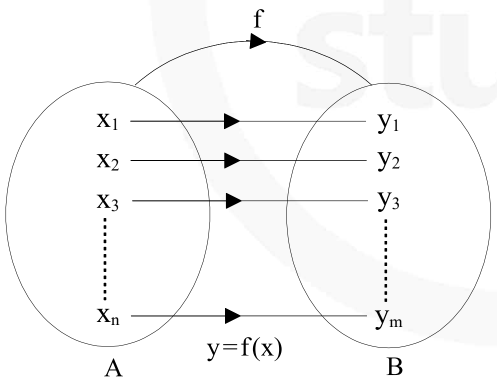
Number of Functions: Since each element of set A can be mapped in m ways

$\Rightarrow$ Number of ways of mapping all n elements of A

$= \left(\underbrace{m\times m\times m\times \ldots\times m}_{n\ \times}\right)$ ways $= m^n$ ways

Conclusion: $2^{nm} \geq m^n\ \forall\ m, n \in N$

NUMBER OF ONE-ONE FUNCTION (INJECTIVE):



Conclusion: ${}^mP_n \leq m^n$ (total number of functions).

NUMBER OF NON-SURJECTIVE FUNCTIONS (INTO FUNCTIONS)

Number of into functions (N) = Number of ways of distributing n different objects into m distinct boxes so that at least one box is empty.

$$\therefore \quad N=\sum_{r=1}^{m} {}^mC_r(-1)^{r-1}(m-r)^n$$

NUMBER OF SURJECTIVE FUNCTIONS

Number of surjective functions = Total number of functions − number of into functions.

$$= m^n - \sum_{r=1}^{m} {}^mC_r(-1)^{r-1}(m-r)^n$$

$$= m^n + \sum_{r=1}^{m} {}^mC_r(-1)^r(m-r)^n$$

$$= \sum_{r=0}^{m} {}^mC_r(-1)^r(m-r)^n$$

Conclusion: In case when $n(A) = n(B)$, the onto functions will be bijection

Number of onto functions = Number of one-one functions

$$= \sum_{r=0}^{n} {}^nC_r(-1)^r(n-r)^n = n!$$

REMARK

(1) If $n(X) < n(Y)$, then after mapping different elements of X to different elements of Y, we are left with at least one element of Y which is not related with any element of X, and hence, there will be no onto function from X to Y, i.e., all the functions from X to Y will be into.

(2) If $n(X) > n(Y)$, then no injective functions can be formed from X to Y as in this case at least one element of Y has to be related to more than one element of X.

(3) If f from X to Y is a bijective functions, then $n(X) = n(Y)$

Example A function $f: A \to B$, such that set "A" and "B" contain four elements each then find

(i) Total number of functions

(ii) Number of one-one functions

(iii) Number of many one functions

(iv) Number of onto functions

(v) Number of into functions

Sol.

(i) I^st element of A can have its image in 4 ways. Similarly, II^nd, III^rd and IV^th can have 4 options for their image each. Hence number of functions = 4^4

(ii) 4 different elements can be matched in 4! ways

(iii) Number of many-one functions

= Total number of functions − number of one-one functions

= $4^4 - 4!$

(iv) Since 4 elements in B are given hence each should be image of atleast one. So number of onto functions = 4!

(v) Number of into functions = $4^4 - 4!$

11. Functional Equations

Example If $f(0)=1, f(1)=2$ & $f(x)=\frac{1}{2}[f(x+1)+f(x+2)]$, find the value of $f(5)$.

Sol. $f(x+2)=2f(x)-f(x+1)$

thus $f(0+2)=f(2)=2f(0)-f(1)=2(1)-2=0$

$f(3)=2f(1)-f(2)=2(2)-0=4$

$f(4)=2f(2)-f(3)=0-4=-4$

$f(5)=2f(3)-f(4)=2(4)-(-4)=12$

12. Binary Operation

DEFINITION

A binary operation * on a set A is a function from set $A\times A$ to A itself. Thus, * associates each pair $(a_1, a_2)\in A\times A$ to a unique element $(a_1 * a_2)$ of A. Thus, domain of a binary operation defined on set A is $A\times A$ and co-domain is A. Range is subset of A.

For example,

Let $A=\{-1,0,1\}$ and * is a function defined as * $(a_1, a_2)=a_1 . a_2; a_1, a_2\in A$

Now we observe,

$*\ (-1,-1)=(-1).(-1)=1\in A;$

$*\ (-1,0)=(-1).(0)=0\in A;$

$*\ (-1,1)=(-1).(1)=-1\in A;$

$*\ (1,1)=(1).(1)=1\in A;$

$*\ (1,0)=(1).(0)=0\in A;$

$*\ (0,0)=(0).(0)=0\in A;$

Thus, * operated to every pair $(a_1, a_2)\in A\times A$ gives us a unique element of A.

Hence, the function * defined in the above example is a binary operation on set A.

PROPERTIES OF BINARY OPERATION * ON A SET A

(i) **Closure Property:** Since binary operation * on a set A is a function from $A\times A$ to A, it obeys closure law, i.e., $a*b\in A\ \forall\ a,b\in A$. Also we say that A is closed with respect to binary operation *.

(ii) **Associativity:** Binary operation * on a set A is said to be associative, if $a*(b*c)=(a*b)*c\ \forall\ a,b,c\in A$.

(iii) **Commutativity:** Binary operation * on a set A is said to be commutative if $a*b=b*a\ \forall\ a,b\in A$.

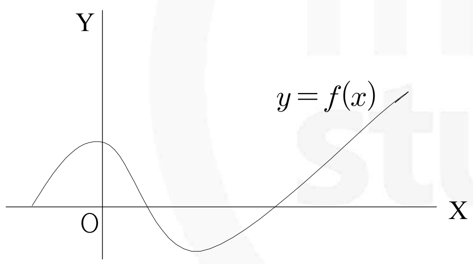
(iv) **Existence of Identity:** An element $e\in A$ is said to be an identity element of set A with respect to binary operation * if $a*e=e*a=a\ \forall\ a\in A$. For example, '+' is a binary operation on set of integer Z. Also $0\in Z$ and $x+0=0+x=x\ \forall\ x\in Z\Rightarrow 0$ is an identity element of set of integers Z with respect to binary operation '+' (addition). Also 0 is called additive identity of set of integers.

(v) **Existence of Inverse:** An element $b \in A$ is said to be inverse of element $b \in A$ with respect to binary operation * if $a * b = e = b * a$; where e is the identity element of A with respect to binary operation *. And we denote $b = a^{-1}$.

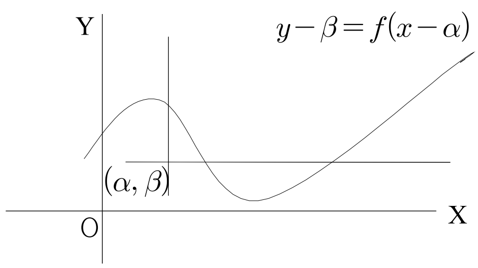
REMARK
1. If a binary operation * on set A is associative and identity element exists in A and every element of A is invertible, then A is said to be a Group with respect to binary operation *.
2. In addition to properties given in remark (1) if * is commutative, then set A is said to be an Abelian Group with respect to binary operation *.
3. If $b = a^{-1}$, then $a = b^{-1}$.
4. Identity element if exists is unique.
5. Inverse of an element if exists is unique provided * is associative.
6. Number of binary operations that can be defined on a set A containing n number of elements is $(n)^{2n}$.

13. Some Graphical Transformation

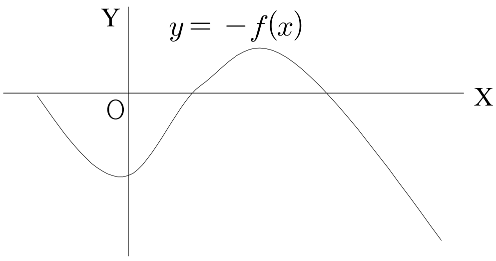
Consider the graph $y = f(x)$ shown alongside.



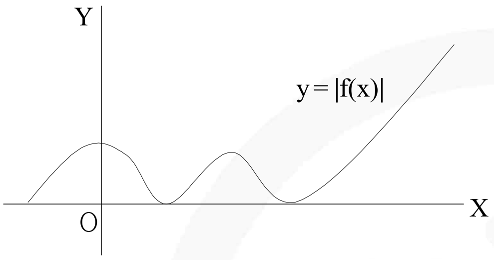
(i) Graph of $y - \beta = f(x - \alpha)$ is drawn by shifting the origin to (α, β) & then translating the graph of $y = f(x)$ w.r.t. new axes



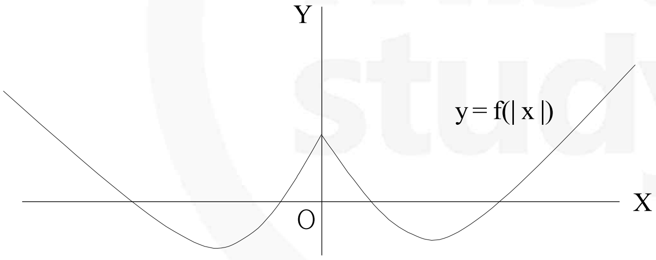
(ii) The graph of $y=-f(x)$ is the mirror image of $f(x)$ in $X-$axis.



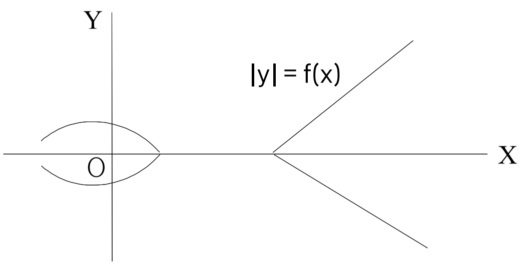
(iii) $y=|f(x)|$ is mirror image of negative portion of $y=f(x)$ in $X-$axis.



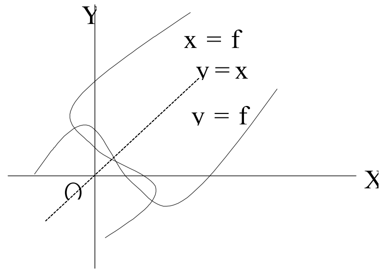
(iv) $y=f(|x|)$ is drawn by taking the mirror image of positive $x-$axis graph in $y-$axis.



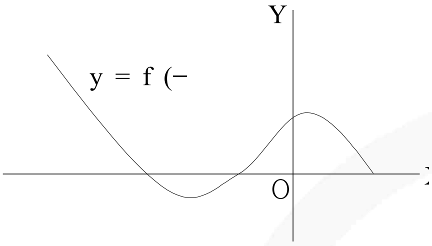
(v) The graph of $|y|=f(x)$ is drawn by deleting those portions of the graph $y=f(x)$ which lie below the $X-$axis and then taking the mirror image of the remaining portion in the $X-$axis, as shown alongside.



(vi) $x = f(y)$ is drawn by taking mirror image of $y = f(x)$ in the line $y = x$.



(vii) $y = f(-x)$ is drawn by taking the mirror image of $y = f(x)$ in $Y-$axis,



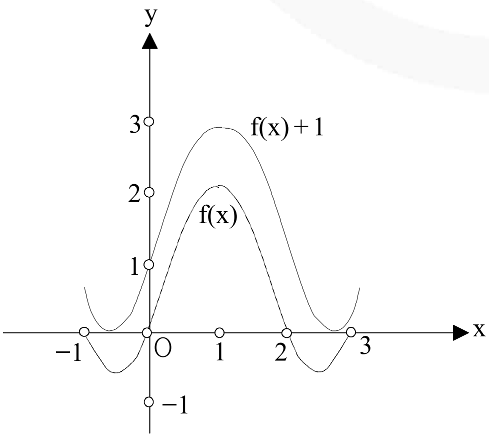
14. Transformation of Graphs

★ **GRAPH OF $y = f(x) + k$:**

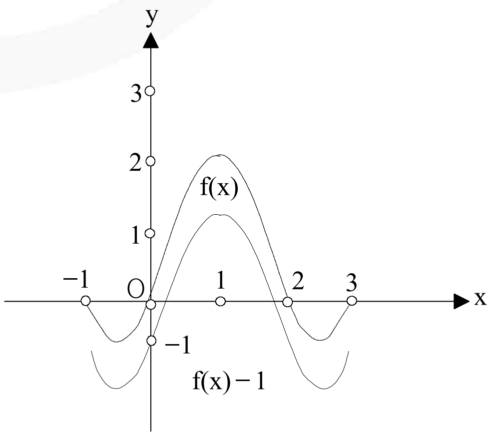
Graph of $y = f(x) + k$ can be obtained by translating graph of $f(x)$ by $|k|$ unit along $y-$axis in the direction same as sing of k, i.e., upward when $k > 0$ and downward when $k < 0$.

REASON:

This is because each out put of the function is added by k. Therefore, each point of graph shifts vertically by k unit.



graph of y = f (x) + 1



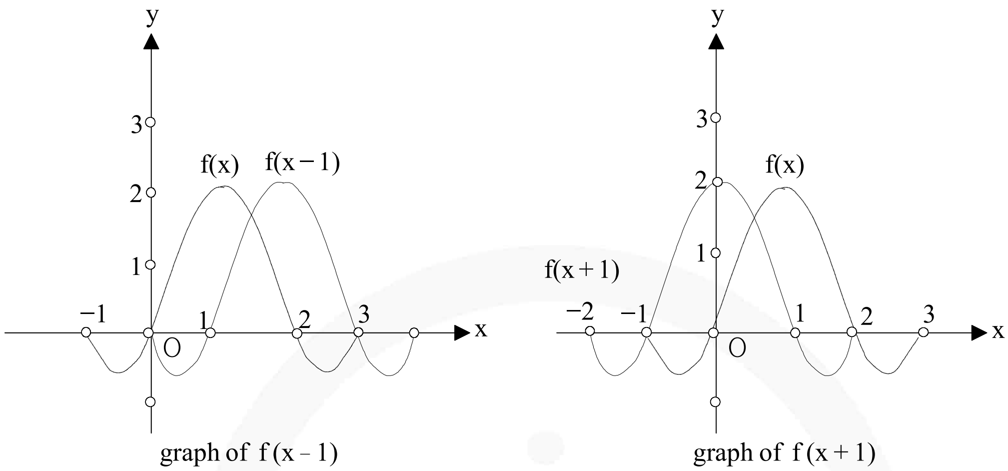
graph of y = f (x) – 1

★ **<u>GRAPH OF $y = f(x + k)$:</u>**

Graph of $y = f(x+k)$ can be obtained by translating graph of $y = f(x)$ by $|k|$ units in the direction opposite to the sign of k along $x-$axis. That is, a addition and subtraction to independent variable leads to horizontal shift.

<u>REASON:</u>

As because each output $f(x)$ of the original function is obtained by the transformed function $f(x+k)$ at the input $x-k$.



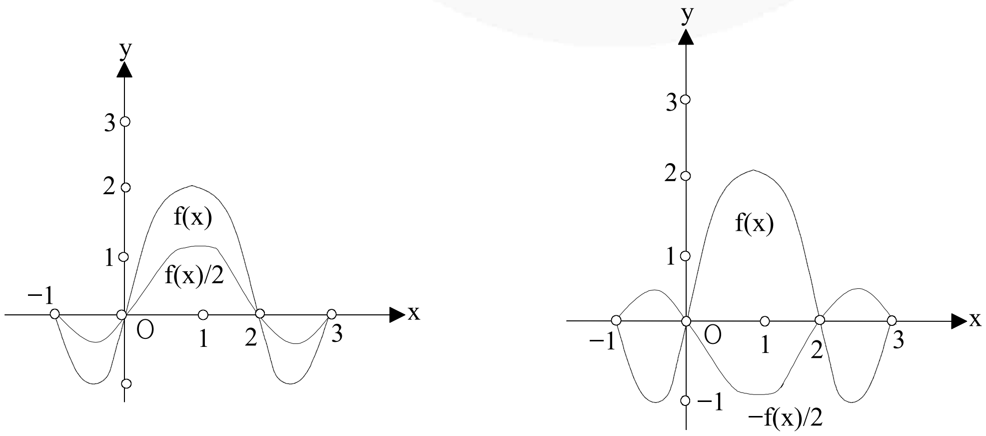
★ **<u>GRAPH OF $y = k(f(x))$:</u>**

Graph of $y = k(f(x))$ can be obtained by vertically stretching or contracting the graph of $f(x)$ depending on the value of k.

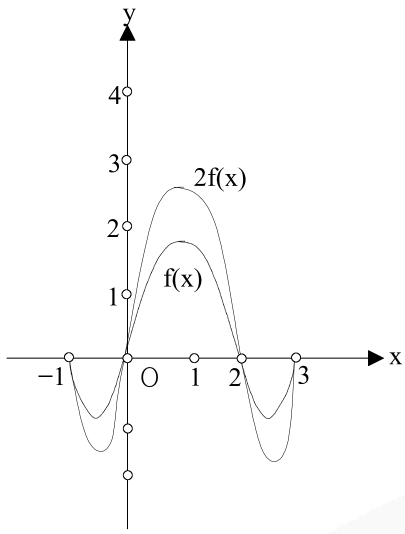
<u>REASON:</u>

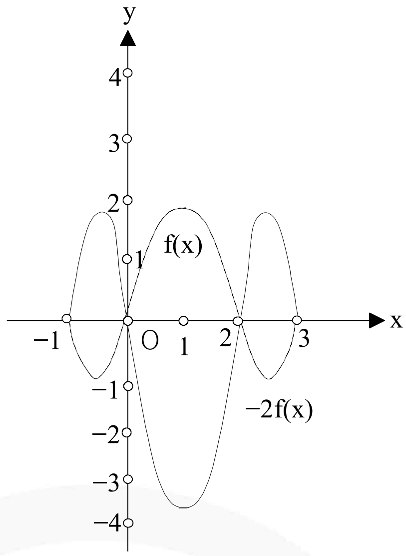
It is because each output of the obtained function becomes k times that of the original function. Hence, due to this transformation no stretching/compression is produce along $x-$axis.

<u>CASE I:</u> When $0 < |k| < 1$



CASE II: When $|k| > 1$

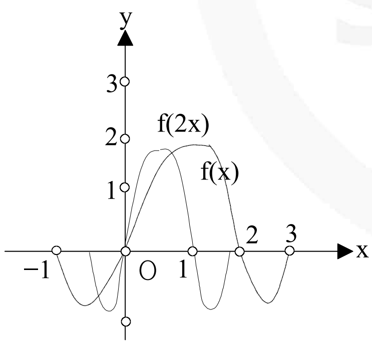




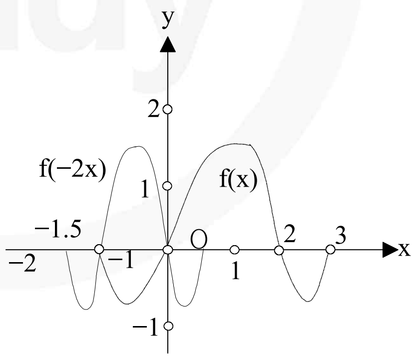
★ **GRAPH OF $y = f(kx)$:**
Graph of $y = f(kx)$ can be obtained by compressing or stretching the graph of $y = f(x)$ along $x-$axis towards $y-$axis or away from $y-$axis depending on the value of k as described below.

CASE I:

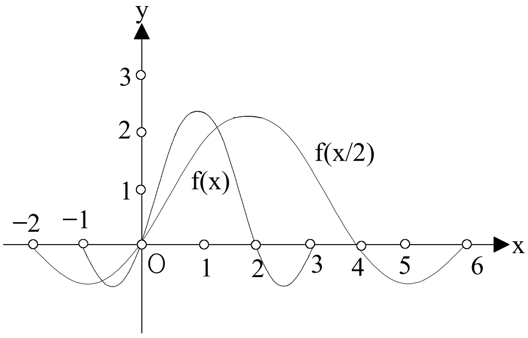
When $|k| > 1$, compressing the graph of $f(x)$ horizontally towards $y-$axis.



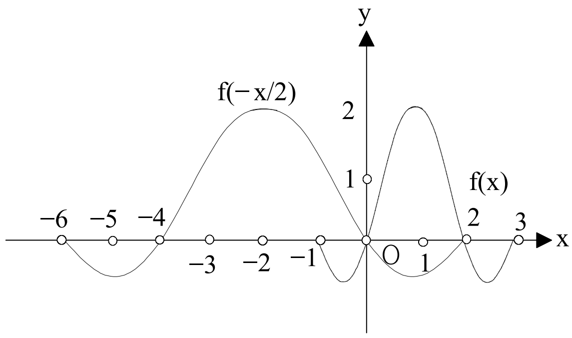
graph of f(2x)



graph of f(−2x)



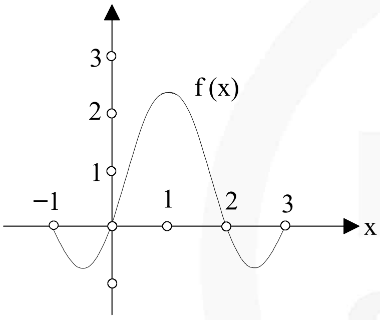
graph of f (x/2)



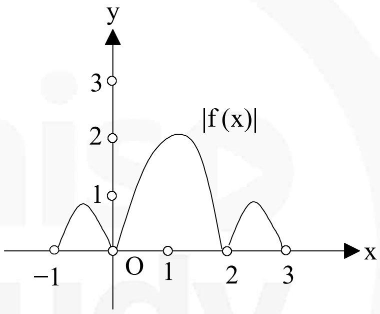
graph of f (-x/2)

★ **GRAPH OF $y = |f(x)|$:**

Graph of $y=|f(x)|$ can be obtained by reflecting the portion of the graph of $f(x)$ laying below $x-$axis on $x-$axis as a mirror and keeping the portion of graph above $x-$axis as it is.



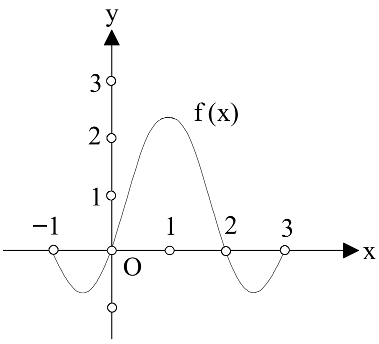
graph of f (x)



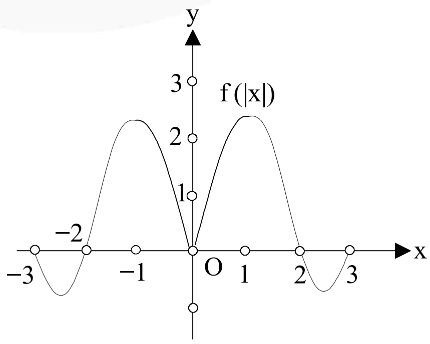
graph of |f (x)|

★ **GRAPH OF $y = f(|x|)$:**

Graph of $y = f(|x|)$ can be obtained by keeping the portion of graph of $f(x)$ on right side of $y-$axis and replacing the portion of the graph of $y=f(x)$ on left side of $y-$axis by the reflection of right graph on $y-$axis.



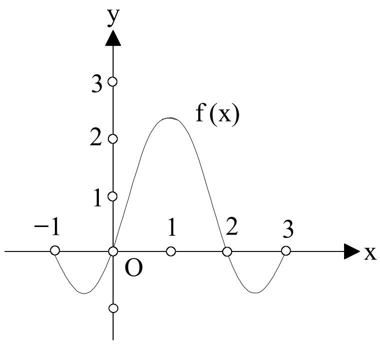
graph of f (x)



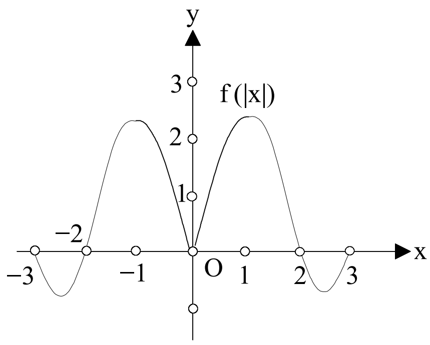
graph of f (|x|)

★ **GRAPH OF $y = f(|x|)$:**

Graph of $y = f(|x|)$ can be obtained by keeping the portion of graph of $f(x)$ on right side of $y-$axis and replacing the portion of the graph of $y = f(x)$ on left side of $y-$axis by the reflection of right graph on $y-$axis.



graph of f (x)



graph of f (|x|)

★ **GRAPH OF $y = |f|x||$ can be obtained in two steps :**

Step 1: Using graph of $y = f(x)$, draw the graph of $f|x|$.

Step 2: Using graph of $y = f|x|$, draw the graph of $y = |f|x||$.

JEE Main Pattern
Exercise (1)

Q1. The function $f: R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$ is-
(a) invertible
(b) injective but not surjective
(c) surjective but not injective
(d) neither injective nor surjective

Q2. If $f_k(x) = 1/k\left(\sin^k x + \cos^k x\right)$, where $x \in R$ and $k \geq 1$, then $f_4(x) - f_6(x)$ is equal to-
(a) $1/6$
(b) $1/3$
(c) $1/4$
(d) $1/12$

Q3. Which of the following relations on R is an equivalence relation?
(a) $aR_1b \Leftrightarrow |a| = |b|$
(b) $aR_2b \Leftrightarrow a \geq b$
(c) $aR_3b \Leftrightarrow a$ divides b
(d) $aR_4b \Leftrightarrow a < b$

Q4. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then, the number of onto functions from E to F is-
(a) 14
(b) 16
(c) 12
(d) 8

Q5. If $g\{f(x)\} = |\sin x|$ and $f\{g(x)\} = (\sin\sqrt{x})^2$, then
(a) $f(x) = \sin^2 x, g(x) = \sqrt{x}$
(b) $f(x) = \sin x, g(x) = |x|$
(c) $f(x) = x^2, g(x) = \sin\sqrt{x}$
(d) f and g cannot be determined

Q6. If S is defined on R by $(x, y) \in S \Leftrightarrow xy \geq 0$. Then S is-

(a) an equivalence relation
(b) reflexive only
(c) symmetric only
(d) transitive only

Q7. Let $f(x) = \frac{\alpha x}{x+1}, x \neq -1$. Then, for what value of α is $f[f(x)] = x$?
(a) $\sqrt{2}$
(b) $-\sqrt{2}$
(c) 1
(d) -1

Q8. If $f(x) = \cos(\log x)$, then $f(x).f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value

(a) -1

(b) $\frac{1}{2}$

(c) -2

(d) None of these

Q9. Let S be the set of all real numbers & let R be a relation on S defined by $aRb \Leftrightarrow |a-b| \leq 1$. Then R is-

(a) Reflexive & Symmetric but not Transitive

(b) Reflexive & Transitive but not Symmetric

(c) Symmetric & Transitive but not Reflexive

(d) an equivalence relation

Q10. If $f(x) = \sin x + \cos x, g(x) = x^2 - 1$, then $g\{f(x)\}$ is invertible in the domain

(a) $\left[0, \frac{\pi}{2}\right]$

(b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(d) $[0, \pi]$

Answer & Solution

ANSWER

Q1	Q2	Q3	Q4	Q5
(c)	(b)	(d)	(a)	(c)
Q6	Q7	Q8	Q9	Q10
(a)	(d)	(a)	(c)	(d)

JEE Advanced Pattern
Exercise (1)

Q1. If $y = f(x) = \dfrac{x+2}{x-1}$, then-
(a) $x = f(y)$
(b) $f(1) = 3$
(c) y increases with x for $x < 1$
(d) f is a rational function of x

Q2. Which of the following functions is periodic?
(a) $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to the real number x
(b) $f(x) = \sin(1/x)$ for $x \neq 0, f(0) = 0$
(c) $f(x) = x\cos x$
(d) None of the above

Q3. Given the relation on $R = \{(a,b),(b,c)\}$ in the set $A = \{a,b,c\}$. Then, the minimum number of ordered pairs which added to R make it an equivalence relation is-
(a) 5
(b) 6
(c) 7
(d) 8

Q4. If $f : [1,\infty) \to [2,\infty)$ is given by $f(x) = x + \dfrac{1}{x}$, then $f^{-1}(x)$ equal.

(a) $\dfrac{x + \sqrt{x^2 - 4}}{2}$
(b) $\dfrac{x}{1 + x^2}$
(c) $\dfrac{x - \sqrt{x^2 - 4}}{2}$
(d) $1 + \sqrt{x^2 - 4}$

Q5. Let $f(x) = \sin\left[\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right]$ for all $x \in R$ and $g(x) = \dfrac{\pi}{2}\sin x$ for all $x \in R$. Let $(fog)(x)$ denotes $f\{g(x)\}$ and $(gof)(x)$ denotes $g\{f(x)\}$. Then, which of the following is/are true?

(a) Range of f is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
(b) Range of fog is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
(c) $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \dfrac{\pi}{6}$
(d) There is an $x \in R$ such that $(gof)(x) = 1$

Q6. Which of the following relations defined on Z is not an equivalence relation-

(a) $(x,y) \in S \Leftrightarrow x \geq y$
(b) $(x,y) \in S \Leftrightarrow x = y$
(c) $(x,y) \in S \Leftrightarrow x - y$ is a multiple of 3
(d) $(x,y) \in S \Leftrightarrow |x - y|$ is even

Q7. If the function $f:[1,\infty)\to[1,\infty)$ is defined by $f(x)=2^{x(x-1)}$, then $f^{-1}(x)$ is-

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$

(b) $\frac{1}{2}\left(1+\sqrt{1+4\log_2 x}\right)$

(c) $\frac{1}{2}\left(1-\sqrt{1+4\log_2 x}\right)$

(d) not defined

Q8. Let $a,b,c\in R$. If $f(x)=ax^2+bx+c$ be such that $a+b+c=3$ and $f(x+y)=f(x)+f(y)+xy$, $\forall x,y\in R$, then $\sum_{n=1}^{10} f(n)$ is equal to-

(a) 330
(b) 165
(c) 190
(d) 255

Q9. If $A=\{1,2,3\}$ then the no. of equivalence relation containing $(1,2)$ is-

(a) 1
(b) 2
(c) 3
(d) 8

Q10. Let f be a one-one function with domain $\{x,y,z\}$ and range $\{1,2,3\}$. It is given that exactly one of the following statements is true and the remaining two are false $f(x)=1, f(y)\neq 1, f(z)\neq 2$. Then $f^{-1}(1)=$

(a) x
(b) y
(c) z
(d) None of these

Answer & Solution

ANSWER

Q1	Q2	Q3	Q4	Q5
(a), (d)	(a)	(a), (c)	(a)	(a), (b), (c)
Q6	Q7	Q8	Q9	Q10
(a)	(b)	(a)	(c)	(b)